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On the singularities of $3-D$ Protter's problem for the wave equation

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ABSTRACT. In this paper we investigate some boundary value problems for the wave equation, which are three-dimensional analogues of Darboux-problems (or Cauchy-Goursat problems) on the plane. These problems have been formulated and studied by M. Protter (1954) in a $3-D$ domain Ω_0, bounded by two characteristic cones and a plane region. Many authors studied these problems using different methods, like: Wiener-Hopf method, special Legendre functions, a priori estimates, nonlocal regularization and others. It is shown that for any $n \in N$ there exists a $C^n(\bar{\Omega}_0)$ - function, for which the corresponding unique generalized solution belongs to $C^n(\bar{\Omega}_0 \backslash O)$, but it has a strong power-type singularity at the point O. This singularity is isolated only at the vertex O of the characteristic cone and does not propagate along the cone. In this paper we investigate the exact behavior of the singular solutions at the point O. Also, we study more general boundary value problems and find that there exist infinite number of smooth right-hand side functions for which the corresponding unique generalized solutions are singular. Finally, some weight a priori estimates are stated.

1. Introduction

Consider the wave equation

$$\Box u \equiv \Delta_x u - u_{tt} = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2} u_{\varphi\varphi} - u_{tt} = f \tag{1.1}$$

in polar or Cartesian coordinates $x_1 = \varrho \cos\varphi$, $x_2 = \varrho \sin\varphi, t$ in a simply connected region $\Omega_0 \subset \mathbb{R}^3$. The region

$$\Omega_0 := \{(x_1, x_2, t) : 0 < t < 1/2, t < \sqrt{x_1^2 + x_2^2} < 1 - t\}$$

is bounded by the disk

$$\Sigma_0 := \{(x_1, x_2, t) : t = 0, x_1^2 + x_2^2 < 1\}$$

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and the characteristic surfaces of (1.1):

$$\Sigma_1 := \{(x_1, x_2, t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = 1 - t\},$$
$$\Sigma_{2,0} := \{(x_1, x_2, t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = t\}.$$

In this work we seek sufficient conditions for the existence and uniqueness of a generalized solution of

Problem $\mathbf{P}_\alpha$. Find a solution of the wave equation (1.1) in Ω_0, which satisfies the boundary conditions

$$P_\alpha : \quad u|_{\Sigma_1} = 0, \quad [u_t + \alpha u]|_{\Sigma_0} = 0, \tag{1.2}$$

where $\alpha \in C(\Sigma_0)$.

The adjoint problem to $\mathbf{P}_\alpha$ is

Problem $\mathbf{P}_\alpha^*$ Find a solution of the wave equation (1.1) in Ω_0 with the boundary conditions:

$$\mathbf{P}_\alpha^* : \quad u|_{\Sigma_{2,0}} = 0, \quad [u_t + \alpha u]|_{\Sigma_0} = 0 \quad . \tag{1.3}$$

The following problems, due to Protter [22], are known as

Protter's Problems. Find a solution of the wave equation (1.1) in Ω_0 with the boundary conditions

$$\begin{array}{llll} P1: & u|_{\Sigma_0 \cup \Sigma_1} = 0, & P1^*: & u|_{\Sigma_0 \cup \Sigma_{2,0}} = 0\,; \\ P2: & u|_{\Sigma_1} = 0, u_t|_{\Sigma_0} = 0, & P2^*: & u|_{\Sigma_{2,0}} = 0, u_t|_{\Sigma_0} = 0\,. \end{array} \tag{1.4}$$

The boundary conditions in problem $P1^*$ (respectively of $P2^*$) are the adjoint boundary conditions to such ones of $P1$ (respectively $P2$) for the equation (1.1) in Ω_0. Protter [22] formulated and investigated problems $P1$ and $P1^*$ in Ω_0 as multidimensional analogues of the Darboux problem on the plane. It is well known that the corresponding Darboux problems in $\mathbb{R}^2$ are well posed, but this is not true for the Protter's problems in $\mathbb{R}^3$. For recent known results concerning the problems (1.4) see papers of Popivanov, Schneider [20], [21] and references therein. For further publications in this area see: [2], [3], [7], [11], [14], [15], [18]. In [1], using Wiener-Hopf techniques for the case $\alpha(\rho) = c/\rho, c \neq 0$, Aldashev studied the Problems P_α and P_α^*. For Problem P_α, which we study in this paper, in [1] he claimed uniqueness of the solution of the class $C^1(\bar{\Omega}_0) \cap C^2(\Omega_0)$, but he did not mention any possible singular solutions.

On the other hand, Bazarbekov [5] gives another analogue of the classical Darboux problem in the same domain Ω_0. Some different statements of Darboux type problems can be found in [4], [6], [13], [16] in bounded or unbounded domains different from Ω_0.

Next, we present here the following well known (see [24], [19])

THEOREM 1.1. *For all $n \in \mathbb{N}, n \geq 4; a_n, b_n$ arbitrary constants, the functions*

$$v_n(\varrho, \varphi, t) = t\varrho^{-n}[\varrho^2 - t^2]^{n-\frac{3}{2}}(a_n \cos n\varphi + b_n \sin n\varphi) \tag{1.5}$$

are classical solutions of the homogeneous problem $P1^*$ and the functions

$$w_n(\varrho, \varphi, t) = \varrho^{-n}[\varrho^2 - t^2]^{n-\frac{1}{2}}(a_n \cos n\varphi + b_n \sin n\varphi) \tag{1.6}$$

are classical solutions of the homogeneous problem $P2^*$.

This theorem shows that for the classical solvability of the problem $P1$ (respectively, $P2$) the function f at least must be orthogonal to all functions (1.5) (respectively,(1.6)). Using Theorem 1.1, Popivanov, Schneider [**21**] proved the existence of some *generalized solutions* of Problems $P1$ and $P2$, which have at least power-type singularities at the vertex (0,0,0) of the cone $\Sigma_{2,0}$. For the homogeneous Problem $P_\alpha{}^*$ (except the case $\alpha \equiv 0$, i.e. except Problem $P2^*$) we do not know solutions analogous to (1.5) and (1.6). Anyway, in the present paper we prove results (see, Theorems 6.1 and 6.2), which ensure the existence of many singular solutions. Here we refer also to Khe Kan Cher [**15**], who gives some nontrivial solutions found for the homogeneous Problems $P1^*$ and $P2^*$, but for the Euler-Poisson-Darboux equation, which are closely connected with the results of Theorem 1.1.

In order to obtain our results, we give the following definition of a solution of Problem P_α with a possible singularity at (0,0,0).

DEFINITION 1.1. *A function $u = u(x_1,x_2,t)$ is called a generalized solution of the problem*

$$P_\alpha : \qquad \Box u = f, \qquad u\big|_{\Sigma_1} = 0, \qquad [u_t + \alpha(x)u]\big|_{\Sigma_0} = 0,$$

in Ω_0, if:

1) $u \in C^1(\bar\Omega_0 \setminus (0,0,0))$, $[u_t + \alpha(x)u]\big|_{\Sigma_0\setminus(0,0,0)} = 0$, $u\big|_{\Sigma_1} = 0$,

2) the identity

$$\int\limits_{\Omega_0} [u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} - fv] dx_1 dx_2 dt = \int\limits_{\Sigma_0} \alpha(x)(uv)(x,0) dx_1 dx_2 \tag{1.7}$$

holds for all $v \in V_0 :=$

$$\{v \in C^1(\bar\Omega_0) : [v_t + \alpha(x)v]\big|_{\Sigma_0} = 0,\ v = 0 \text{ in a neighbourhood of } \Sigma_{2,0}\}.$$

In order to deal successfully with the encountered difficulties, as are the singularities on the cone $\Sigma_{2,0}$, we introduce the region

$$\Omega_\varepsilon = \Omega_0 \cap \{\rho - t > \varepsilon\}, \varepsilon \in [0, 1),$$

which in polar coordinates becomes

$$\Omega_\varepsilon = \{(\varrho, \varphi, t) : t > 0,\ 0 \le \varphi < 2\pi,\ \varepsilon + t < \varrho < 1 - t\}. \tag{1.8}$$

and we define the notion of a *generalized solution* of Problem P_α in $\Omega_\varepsilon, \varepsilon \in (0,1)$ (see Definition 2.1). Note that, if a generalized solution u belongs to $C^1(\bar\Omega_\varepsilon) \cap C^2(\Omega_\varepsilon)$, it is called a *classical solution* of Problem P_α in Ω_ε, $\varepsilon \in (0,1)$, and it satisfies the wave equation (1.1) in Ω_ε. It should be pointed out that the case $\varepsilon = 0$ is totally different from the case $\varepsilon \neq 0$.

This paper, besides Introduction, consists of five more sections. In Section 2, using some appropriate technics, we formulate the $2-D$ boundary problems $P_{\alpha,1}$ and $P_{\alpha,2}$, corresponding to the $3-D$ Problem P_α. The aim of Section 3 is to treat Problem $P_{\alpha,2}$. For this reason, we construct and study the integral equation assigned to the under consideration wave equation of general form. Also we present results concerning the classical solutions of Problem $P_{\alpha,2}$ in $\Omega_\varepsilon, \varepsilon \in (0,1)$ and give corresponding a priori estimates. In Section 4 we prove Theorems 4.1 and 4.2 which ensure the existence and uniqueness of a generalized solution of Problem $P_{\alpha,1}$ in $\Omega_\varepsilon, \varepsilon \in [0,1)$. Using the results of the previous section, in Section 5 we study the existence and uniqueness of a generalized solution of $3-D$ Problem P_α. More precisely, Theorem 5.1 ensure the uniqueness of a generalized solution of problem P_α in $\Omega_\varepsilon, \varepsilon \in [0,1)$, while Theorems 5.2 and 5.3 ensure the existence of a generalized solution, satisfying corresponding a priori estimates for problem P_α in the case, where the right-hand side of the wave equation is a trigonometric polynomial or trigonometric series. Finally, in Section 6 we present some singular generalized solutions which are smooth enough away from the point $(0,0,0)$, while at the point $(0,0,0)$ they have power-type singularity of the class ρ^{-n}. Precisely, in Theorem 6.1 we prove the following result:

Let $\alpha \geq 0$ and $\alpha \in C^\infty$. Then for each $n \in N$, $n \geq 4$, there exists a function $f_n(\varrho,\varphi,t) \in C^{n-2}(\bar{\Omega}_0)$, for which the corresponding general solution u_n of the problem P_α belongs to $C^n(\bar{\Omega}_0\backslash(0,0,0))$ and satisfies the estimate

$$|u_n(\rho,\varphi,\rho)| \geq \rho^{-n}|\cos n\varphi|, \qquad 0<\rho<1. \tag{1.9}$$

When $\alpha \equiv 0$ the upper estimate holds, and in this case we have also the following two-sided estimate

$$\rho^{-n}|\cos n\varphi| \leq |u_n(\rho,\varphi,\rho)|, \qquad |u_n(\rho,\varphi,0)| \leq C_2\rho^{-n}|\cos n\varphi|, \tag{1.10}$$

with $C_2 = const$. That is, in the case of Problem P2 the exact behavior of $u_n(x_1,x_2,t)$ around (0,0,0) is $(x_1^2+x_2^2)^{-n/2}$.

REMARK 1.1. *In Theorem 6.2 we find some different singular solutions for the same problem P_α. It is particularly interesting that for any parameter $\alpha(x) \geq 0$, involved in the boundary condition (1.2) on Σ_0, there are infinitely many singular solutions of the wave equation. Note, that all these solutions have strong singularities at the vertex $(0,0,0)$ of the cone $\Sigma_{2,0}$. These singularities of generalized solutions do not propagate in the direction of the bicharacteristics on the characteristic cone. It is traditionally assumed that the wave equation with right-hand side sufficiently smooth in $\bar{\Omega}_0$ cannot have a solution with an isolated singular point. For results concerning the propagation of singularities for second order operators, see Hörmander* [**10**], *Chapter 24.5. For some related results in the case of plane Darboux-Problem, see* [**17**].

REMARK 1.2. *In 1960 Garabedian proved* [**8**] *the uniqueness of classical solution of Problem P1. Existence and uniqueness results for a generalized solution of Problems P1 and P2 can be found in* [**20**], [**21**].

REMARK 1.3. *Considering Problems P1 and P2, Popivanov, Schneider* [**19**] *announced the existence of singular solutions of both wave and degenerate hyperbolic equation. The proofs of that results are given in* [**21**] *and* [**20**] *respectively. First a priori estimates for singular solutions of Protter's Problems P1 and P2, concerning the wave equation in $\mathbb{R}^3$, were obtained in* [**21**]. *In* [**2**] *Aldashev mentions the results of* [**19**] *and, for the case of the wave equation in $\mathbb{R}^{m+1}$, he shows that there exist solutions of Problem P1 (respectively, P2) in the domain Ω_ε, which grow up on the cones $\Sigma_{2,\varepsilon}$ like $\varepsilon^{-(n+m-2)}$ (respectively, $\varepsilon^{-(n+m-1)}$), when for $\varepsilon \to 0$ the cones $\Sigma_{2,\varepsilon} := \{\rho = t+\varepsilon\}$ approximate $\Sigma_{2,0}$. It is obvious that for $m=2$ this results can be compared with the estimate (1.10) of Theorem 6.1 and the analogous estimate of Theorem 6.2. More comments, concerning Aldashev's results* [**2**], *we give in Section 6. Finally, we point out that in the case of an equation, which involves the wave operator and nonzero lower terms, Karatoprakliev* [**12**] *obtained a priori estimates for the smooth solutions of Problem P1 in Ω_0.*

2. Preliminaries

In this section we consider the wave equation (1.1) in a simply connected region

$$\Omega_\varepsilon := \{(\varrho,\varphi,t) : 0 < t < (1-\varepsilon)/2,\ 0 \le \varphi < 2\pi,\ \varepsilon + t < \varrho < 1-t\}, \tag{2.1}$$

bounded by the disc $\Sigma_0 := \{(\varrho,\varphi,t) : t = 0, \varrho < 1\}$ and the characteristic surfaces of (1.1)

$$\Sigma_1 := \{(\varrho,\varphi,t) : 0 \le \varphi < 2\pi, \varrho = 1-t\},$$
$$\Sigma_{2,\varepsilon} := \{(\varrho,\varphi,t) : 0 \le \varphi < 2\pi, \varrho = \varepsilon + t\}.$$

We seek sufficient conditions for the existence and uniqueness of a generalized solution of the equation (1.1) with $f \in C(\bar{\Omega}_\varepsilon)$, which satisfies the following boundary conditions:

$$P_\alpha : \qquad u\big|_{\Sigma_1 \cap \partial\Omega_\varepsilon} = 0, \qquad [u_t + \alpha(\rho)u]\big|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0; \tag{2.2}$$

$$P^*_\alpha : \qquad u\big|_{\Sigma_{2,\varepsilon}} = 0, \qquad [u_t + \alpha(\rho)u]\big|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0, \tag{2.3}$$

where for the sake of simplicity, we set $\alpha(x) \equiv \alpha(|x|) = \alpha(\rho) \in C^1([0,1])$. The problem P^*_α is the adjoint problem to Problem P_α in Ω_ε.

Now, to obtain our results we define the notion of a generalized solution as follows.

DEFINITION 2.1. *A function $u = u(\varrho,\varphi,t)$ is called a generalized solution of Problem P_α in Ω_ε, $\varepsilon > 0$, if:*
1) $u \in C^1(\bar{\Omega}_\varepsilon)$, $u\big|_{\Sigma_1\cap\partial\Omega_\varepsilon} = 0$; $[u_t + \alpha(\varrho)u]\big|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0$,
2) the identity

$$\int_{\Omega_\varepsilon} [u_t v_t - u_\varrho v_\varrho - \frac{1}{\varrho^2} u_\varphi v_\varphi - fv]\varrho\, d\varrho\, d\varphi\, dt = \int_{\Sigma_0\cap\partial\Omega_\varepsilon} \varrho\alpha(\varrho) uv\, d\varrho d\varphi \tag{2.4}$$

holds for all

$$v \in V_\varepsilon := \{v \in C^1(\bar{\Omega}_\varepsilon) : [v_t + \alpha(\varrho)v]\big|_{\Sigma_0} = 0, v\big|_{\Sigma_{2,\varepsilon}} = 0\}$$

The following proposition describes the properties of generalized solutions of Problem P_α in Ω_ε.

LEMMA 2.1. *Each generalized solution of Problem P_α in Ω_0 is also a generalized solution of the same problem in Ω_ε for $\varepsilon > 0$.*

In view of (1.7), the equality (2.4) holds for each function $v \in V_0$ with the property $v \equiv 0$ in $\Omega_0 \setminus \Omega_\varepsilon$. To approximate an arbitrary function $v_1 \in V_\varepsilon$ by such functions in $W^1_2(\Omega_\varepsilon)$ we make the following steps:

Step 1. Setting $v_2(\varrho,\varphi,t) = e^{t\alpha(\varrho)}v_1(\varrho,\varphi,t)$, we get

$$\frac{\partial v_2}{\partial t}\big|_{\Sigma_0} = 0, \qquad v_2\big|_{\Sigma_{2,\varepsilon}} = 0 \tag{2.5}$$

Step 2. The function $v_2(\varrho,\varphi,t)$ could be approximated in $W^1_2(\Omega_\varepsilon)$ by some functions, which satisfy (2.5) and are zero in a neighborhood of the circle

$\{\varrho=\varepsilon, t=0\}$. In fact, such functions are:

$$v_{2m}(\varrho,\varphi,t) := v_2(\varrho,\varphi,t)\psi(m\sqrt{(\varrho-\varepsilon)^2+t^2}), \quad m\to\infty,$$

where $\psi\in C^\infty(R^1)$, $\psi(s)=0$, for $s\leq 1$ and $\psi(s)=1$, for $s>2$.

Step 3. Each function $v_{2m}(\varrho,\varphi,t)$ could be approximated in $W_2^1(\Omega_\varepsilon)$ by some functions, which satisfy (2.5), and are zero in a neighborhood of the cone $\{\varrho=t+\varepsilon\}$:

$$v_k(\varrho,\varphi,t) := v_{2m}(\varrho,\varphi,t)\psi((t-\varrho+\varepsilon)k), \quad k\to\infty.$$

In the special, but main case, when

$$f(\varrho,\varphi,t) = f_n^{(1)}(\varrho,t)\cos n\varphi + f_n^{(2)}(\varrho,t)\sin n\varphi \tag{2.6}$$

we ask the generalized solution to be of the form

$$u(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi + u_n^{(2)}(\varrho,t)\sin n\varphi \tag{2.7}$$

If we introduce the function $u^{(1)}(\varrho,t) := \begin{cases} u_n^{(1)} & \text{for } f^{(1)}=f_n^{(1)}, \\ u_n^{(2)} & \text{for } f^{(1)}=f_n^{(2)}, \end{cases}$

then, in view of (1.1), we conclude that

$$\tilde{\Box}u^{(1)} = \frac{1}{\varrho}(\varrho u_\varrho^{(1)})_\varrho - \frac{n^2}{\varrho^2}u^{(1)} - u_{tt}^{(1)} = f^{(1)} \tag{2.8}$$

in $G_\varepsilon = \{(\varrho,t): t>0, \varepsilon+t<\varrho<1-t\}$, which is bounded by the sets:

$$\begin{aligned} &S_0 = \{(\varrho,t): t=0, 0<\varrho<1\}, \\ &S_1 = \{(\varrho,t): \varrho=1-t\}, \quad S_{2,\varepsilon} = \{(\varrho,t): \varrho=t+\varepsilon\}. \end{aligned} \tag{2.9}$$

Instead of the equation (2.8), consider the more general equation

$$Lu^{(1)} = \frac{1}{\varrho}(\varrho u_\varrho^{(1)})_\varrho - u_{tt}^{(1)} + d(\rho,t)u^{(1)} = f^{(1)}, \tag{2.10}$$

with the same boundary conditions. In this case, the two-dimensional problem corresponding to P_α is

$$P_{\alpha,1}: \begin{cases} Lu^{(1)} = f^{(1)} \text{ in } G_\varepsilon, \\ u^{(1)}|_{S_1} = 0, \qquad [u_t^{(1)} + \alpha(\varrho)u^{(1)}]|_{S_0\setminus(0,0)} = 0 \end{cases} \tag{2.11}$$

and its generalized solution is defined by

Definition 2.2. *A function $u^{(1)} = u^{(1)}(\varrho,t)$ is called a generalized solution of problem $P_{\alpha,1}$ in G_ε, $\varepsilon>0$, if:*

1) $u\in C^1(\bar{G}_\varepsilon)$, $[u_t+\alpha(\varrho)u]|_{S_0\cap\partial G_\varepsilon}=0$, $u|_{S_1\cap\partial G_\varepsilon}=0$;

2) the identity

$$\int_{G_\varepsilon}[u_t^{(1)}v_t - u_\varrho^{(1)}v_\varrho + d(\varrho,t)u^{(1)}v - f^{(1)}v]\varrho d\varrho\, dt = \int_{S_0\cap\partial G_\varepsilon} \varrho\alpha(\varrho)u^{(1)}v\, d\varrho \tag{2.12}$$

holds for all

$$v\in V_\varepsilon^{(1)} = \{v\in C^1(\bar{G}_\varepsilon): [v_t+\alpha(\rho)v]|_{S_0}=0, v|_{S_{2,\varepsilon}}=0\}.$$

By introducing a new function

$$u^{(2)}(\varrho,t)=\varrho^{\frac{1}{2}}u^{(1)}(\varrho,t), \tag{2.13}$$

we transform (2.10) to the equation

$$u^{(2)}_{\varrho\varrho}-u^{(2)}_{tt}+\left[d(\varrho,t)+\frac{1}{4\varrho^2}\right]u^{(2)}=\varrho^{\frac{1}{2}}f^{(1)}, \tag{2.14}$$

with the string operator in the main part. Substituting the new coordinates

$$\xi=1-\varrho-t,\eta=1-\varrho+t, \tag{2.15}$$

from (2.14) we derive

$$U_{\xi\eta}+\frac{1}{4}\left[d^{(2)}(\xi,\eta)+(2-\eta-\xi)^{-2}\right]U=\frac{1}{4\sqrt{2}}(2-\eta-\xi)^{\frac{1}{2}}F(\xi,\eta), \tag{2.16}$$

in $D_\varepsilon=\{(\xi,\eta):0<\xi<\eta<1-\varepsilon\}$, where

$$U(\xi,\eta)=u^{(2)}(\rho(\xi,\eta),t(\xi,\eta)),\qquad F(\xi,\eta)=f^{(1)}(\rho(\xi,\eta),t(\xi,\eta)). \tag{2.17}$$

Thus, we reduced the problem $P_{\alpha,1}$ to the Darboux-Goursat problem for the more general equation (2.10) with the same boundary conditions:

$$P_{\alpha,2}:\ \left\{\begin{array}{c} U_{\xi\eta}+c(\xi,\eta)U=g(\xi,\eta) \text{ in } D_\varepsilon,\\ U(0,\eta)=0,(U_\eta-U_\xi)(\xi,\xi)+\alpha(1-\xi)U(\xi,\xi)=0.\end{array}\right. \tag{2.18}$$

In view of the above observations, the wave equation (1.1) transforms finally to the equation

$$U_{\xi\eta}+\frac{1-4n^2}{4(2-\xi-\eta)^2}U=\frac{1}{4\sqrt{2}}(2-\eta-\xi)^{\frac{1}{2}}F(\xi,\eta), \tag{2.19}$$

which is of the form (2.16).

3. The integral equation corresponding to Problem $P_{\alpha,2}$

Set

$$\begin{aligned} c(\xi,\eta)&=\frac{1-4n^2}{4(2-\xi-\eta)^2}\in C^\infty(\bar{D}_0\setminus(1,1)),\\ g(\xi,\eta)&=\frac{1}{4\sqrt{2}}(2-\xi-\eta)^{\frac{1}{2}}F(\xi,\eta).\end{aligned} \tag{3.1}$$

Then the equation (2.19), in new therms, takes the form of the equation in (2.18). Remark, that if $f_n^{(i)}\in C^0(\bar{G}_0)$, $i=1,2,$ then $g\in C(\bar{D}_0)$, while $f_n^{(i)}\in C^k(\bar{G}_0)$, $i=1,2,$ then $g\in C^k(\bar{D}_0\backslash(1,1))$.

In order to investigate the smoothness and the singularity of a solution of the original $3-D$ problem P_α on $\Sigma_{2,0}$, we are seeking for a classical solution of the corresponding $2-D$ problem $P_{\alpha,2}$ not only in the domain D_ε, but also in the domain

$$D_\varepsilon^{(1)}:=\{(\xi,\eta):0<\xi<\eta<1,0<\xi<1-\varepsilon\},\quad \varepsilon>0. \tag{3.2}$$

Clearly, $D_\varepsilon\subset D_\varepsilon^{(1)}$.

Consider now the equation from (2.18), i.e.

$$U_{\xi\eta}+c(\xi,\eta)U=g(\xi,\eta) \text{ in } D_\varepsilon^{(1)}, \tag{3.3}$$

where $c(\xi,\eta)\in C(\bar{D}_\varepsilon^{(1)})$, $g(\xi,\eta)\in C(\bar{D}_\varepsilon^{(1)})$, $\varepsilon>0$.

Next, for any $(\xi_0, \eta_0) \in D_\varepsilon^{(1)}$, we consider the sets

$$\Pi := \{(\xi, \eta) : 0 < \xi < \xi_0, \xi_0 < \eta < \eta_0\}, \quad T := \{(\xi, \eta) : 0 < \xi < \eta, 0 < \eta < \xi_0\}$$

and we construct an equivalent integral equation to the problem $P_{\alpha,2}$, in such a way that any solution of the problem $P_{\alpha,2}$ to be also a solution of the constructed integral equation. For this reason, we consider the following integrals:

$$\begin{aligned} I_0 &:= \iint_\Pi [g(\xi, \eta) - c(\xi, \eta)U(\xi, \eta)]\, d\eta\, d\xi = \int_0^{\xi_0} \int_{\xi_0}^{\eta_0} U_{\xi\eta}(\xi, \eta)\, d\eta\, d\xi \\ &= \int_0^{\xi_0} [U_\xi(\xi, \eta_0) - U_\xi(\xi, \xi_0)]\, d\xi = U(\xi_0, \eta_0) - U(\xi_0, \xi_0) \end{aligned}$$

and

$$\begin{aligned} I_1 &:= \iint_T [g(\xi, \eta) - c(\xi, \eta)U(\xi, \eta)]\, d\eta\, d\xi = \int_0^{\xi_0} \int_{\xi}^{\xi_0} U_{\xi\eta}(\xi, \eta)\, d\eta\, d\xi \\ &= \int_0^{\xi_0} [U_\xi(\xi, \xi_0) - U_\xi(\xi, \xi)]\, d\xi = U(\xi_0, \xi_0) - \int_0^{\xi_0} U_\xi(\xi, \xi)\, d\xi. \end{aligned}$$

On the other side,

$$I_1 = \int_0^{\xi_0} \int_0^{\eta} U_{\xi\eta}(\xi, \eta)\, d\xi\, d\eta = \int_0^{\xi_0} U_\eta(\eta, \eta)\, d\eta.$$

Hence, we see that:

$$\begin{aligned} 2I_1 &= U(\xi_0, \xi_0) + \int_0^{\xi_0} [U_\eta(\xi, \xi) - U_\xi(\xi, \xi)]\, d\xi \\ &= U(\xi_0, \xi_0) - \int_0^{\xi_0} \alpha(1 - \xi)U(\xi, \xi)\, d\xi, \\ I_0 + 2I_1 &= U(\xi_0, \eta_0) - \int_0^{\xi_0} \alpha(1 - \xi)U(\xi, \xi)\, d\xi. \end{aligned}$$

From the latest relation we obtain

$$\begin{aligned} U(\xi_0, \eta_0) = &\int_0^{\xi_0} \int_{\xi_0}^{\eta_0} [g(\xi, \eta) - c(\xi, \eta)U(\xi, \eta)]\, d\eta\, d\xi \\ &+ 2\int_0^{\xi_0} \int_0^{\eta} [g(\xi, \eta) - c(\xi, \eta)U(\xi, \eta)]\, d\xi\, d\eta \qquad (3.4) \\ &+ \int_0^{\xi_0} \alpha(1 - \xi)U(\xi, \xi)\, d\xi, \text{ for } (\xi_0, \eta_0) \in \bar{D}_\varepsilon^{(1)}, \end{aligned}$$

which is the desired integral equation.

Next, we set

$$M_g := \sup_{D_\varepsilon^{(1)}} |g(\xi, \eta)|, \quad c(\varepsilon) := \sup_{D_\varepsilon^{(1)}} |c(\xi, \eta)|, \quad M_\alpha := \sup_{[0,1]} |\alpha(\xi)| \qquad (3.5)$$

and state the following

THEOREM 3.1. *Let $c(\xi,\eta) \in C(\bar{D}^{(1)}_\varepsilon)$, $g(\xi,\eta) \in C(\bar{D}^{(1)}_\varepsilon)$, $\varepsilon > 0$. Then there exists a classical solution $U(\xi,\eta) \in C^1(\bar{D}^{(1)}_\varepsilon)$ of the equation (3.3) which satisfies the boundary conditions (2.18) with $U_{\xi\eta}(\xi,\eta) \in C(\bar{D}^{(1)}_\varepsilon)$ and*

$$\text{(3.6)}\qquad \begin{aligned} &|U(\xi_0,\eta_0)| \le \xi_0 M_g[c(\varepsilon)+M_\alpha]^{-1}\exp[c(\varepsilon)+M_\alpha] \quad \text{in } D^{(1)}_\varepsilon,\\ &\sup_{D^{(1)}_\varepsilon}\{|U_\xi|,|U_\eta|\} \le M_g[c(\varepsilon)+M_\alpha]^{-1}\exp[c(\varepsilon)+2M_\alpha]. \end{aligned}$$

Proof. In order to solve the integral equation (3.4), we use the following sequence of successive approximations $U^{(n)}$, defined by the formula

$$\text{(3.7)}\qquad \begin{aligned} U^{(n+1)}(\xi_0,\eta_0) = &\int_0^{\xi_0}\int_{\xi_0}^{\eta_0}[g(\xi,\eta)-c(\xi,\eta)U^{(n)}(\xi,\eta)]\,d\eta\,d\xi\\ &+2\int_0^{\xi_0}\int_0^{\eta}[g(\xi,\eta)-c(\xi,\eta)U^{(n)}(\xi,\eta)]\,d\xi\,d\eta\\ &+\int_0^{\xi_0}\alpha(1-\xi)U^{(n)}(\xi,\xi)\,d\xi,\\ U^{(0)}(\xi_0,\eta_0) = 0,&\quad \text{in } D^1_\varepsilon. \end{aligned}$$

We will show that for any $(\xi_0,\eta_0) \in \bar{D}^{(1)}_\varepsilon$ and $n \in \mathbb{N}$ it holds

$$\text{(3.8)}\qquad |(U^{(n+1)}-U^{(n)})(\xi_0,\eta_0)| \le \frac{M_g[c(\varepsilon)+M_\alpha]^n\xi_0^{n+1}}{(n+1)!}.$$

Indeed:

$$1)\quad U^{(1)}(\xi_0,\eta_0) = \int_0^{\xi_0}\int_{\xi_0}^{\eta_0} g(\xi,\eta)\,d\eta\,d\xi + 2\int_0^{\xi_0}\int_0^{\eta} g(\xi,\eta)\,d\xi\,d\eta,$$

and hence

$$|U^{(1)}(\xi_0,\eta_0)| \le M_g[\xi_0(\eta_0-\xi_0)+\xi_0^2] = M_g\xi_0\eta_0 \le M_g\xi_0.$$

2) Let, by the induction hypothesis (3.8),

$$|(U^{(n)}-U^{(n-1)})(\xi_0,\eta_0)| \le \frac{M_g}{n!}[c(\varepsilon)+M_\alpha]^{n-1}\xi_0^n := A_n\xi_0^n$$

be satisfied. Then, it follows that

$$\begin{aligned} |(U^{(n+1)}-U^{(n)})(\xi_0,\eta_0)| = \Big|-&\int_0^{\xi_0}\int_{\xi_0}^{\eta_0} c(\xi,\eta)(U^{(n)}-U^{(n-1)})(\xi,\eta)\,d\eta\,d\xi\\ -2\int_0^{\xi_0}\int_0^{\eta} c(\xi,\eta)(U^{(n)}-&U^{(n-1)})(\xi,\eta)\,d\xi\,d\eta + \int_0^{\xi_0}\alpha(1-\xi)(U^{(n)}-U^{(n-1)})(\xi,\xi)\,d\xi\Big|\\ &\le A_n\Big[c(\varepsilon)\Big(\int_0^{\xi_0}\int_{\xi_0}^{\eta_0}\xi^n\,d\eta\,d\xi + 2\int_0^{\xi_0}\int_0^{\eta}\xi^n\,d\xi\,d\eta\Big) + M_\alpha\int_0^{\xi_0}\xi^n\,d\xi\Big]\\ &= A_n\Big[c(\varepsilon)\Big(\frac{1}{n+1}\xi_0^{n+1}(\eta_0-\xi_0) + \frac{2}{(n+1)(n+2)}\xi_0^{n+2}\Big) + \frac{M_\alpha}{n+1}\xi_0^{n+1}\Big] \end{aligned}$$

$$= A_n\left[c(\varepsilon)\left(\frac{1}{n+1}\xi_0^{n+1}\eta_0 - \frac{n}{(n+1)(n+2)}\xi_0^{n+2}\right) + \frac{M_\alpha}{n+1}\xi_0^{n+1}\right]$$

$$\leq A_n\left[\frac{c(\varepsilon)}{n+1}\xi_0^{n+1} + \frac{M_\alpha}{n+1}\xi_0^{n+1}\right] = \frac{M_g}{(n+1)!}[c(\varepsilon)+M_\alpha]^n\xi_0^{n+1} = A_{n+1}\xi_0^{n+1}.$$

So, the inequality (3.8) is proved and hence the uniform convergence of the sequence $\{U^{(m)}(\xi,\eta)\}_{m\in\mathbb{N}}$ in $\bar{D}_\varepsilon^{(1)}$ is obvious. For the limit function $U \in C(\bar{D}_\varepsilon^{(1)})$ we obtain the integral equality (3.4) and $U(0,\eta_0) = 0$.

Also, in view of (3.8), we see that

$$|(U^{(n+1)}(\xi_0,\eta_0)| = \left|\sum_{k=0}^{n}(U^{(k+1)} - U^{(k)})(\xi_0,\eta_0)\right| \leq \xi_0 M_g \sum_{k=0}^{n}\frac{[c(\varepsilon)+M_\alpha]^k}{(k+1)!}\xi_0^k$$

$$\leq \xi_0 M_g[c(\varepsilon)+M_\alpha]^{-1}\exp[c(\varepsilon)+M_\alpha],$$

and therefore

$$|U(\xi_0,\eta_0)| \leq \xi_0 M_g[c(\varepsilon)+M_\alpha]^{-1}\exp[c(\varepsilon)+M_\alpha].$$

To estimate the first derivatives of the function U, by (3.7), we get:

$$U_{\xi_0}^{(n+1)}(\xi_0,\eta_0) = \alpha(1-\xi_0)U^{(n)}(\xi_0,\xi_0) \tag{3.9}$$

$$+\int_0^{\xi_0}[g(\xi,\xi_0) - c(\xi,\xi_0)U^{(n)}(\xi,\xi_0)]\,d\xi + \int_{\xi_0}^{\eta_0}[g(\xi_0,\eta) - c(\xi_0,\eta)U^{(n)}(\xi_0,\eta)]\,d\eta,$$

and

$$U_{\eta_0}^{(n+1)}(\xi_0,\eta_0) = \int_0^{\xi_0}[g(\xi,\eta_0) - c(\xi,\eta_0)U^{(n)}(\xi,\eta_0)]\,d\xi. \tag{3.10}$$

Using (3.8) and (3.9) we see that

$$|U_{\xi_0}^{(1)}(\xi_0,\eta_0)| = \left|\int_0^{\xi_0} g(\xi,\xi_0)\,d\xi + \int_{\xi_0}^{\eta_0} g(\xi_0,\eta)\,d\eta\right|$$

$$\leq M_g(\xi_0+\eta_0-\xi_0) = M_g\eta_0 \leq M_g,$$

and

$$|(U_{\xi_0}^{(n+1)} - U_{\xi_0}^{(n)})(\xi_0,\eta_0)| = \left|-\int_0^{\xi_0} c(\xi,\xi_0)(U^{(n)} - U^{(n-1)})(\xi,\xi_0)]\,d\xi\right.$$

$$\left.-\int_{\xi_0}^{\eta_0} c(\xi_0,\eta)(U^{(n)} - U^{(n-1)})(\xi_0,\eta)\,d\eta + \alpha(1-\xi_0)(U^{(n)} - U^{(n-1)})(\xi_0,\xi_0)\right|$$

$$\leq \frac{M_g}{n!}[c(\varepsilon)+M_\alpha]^{n-1}\left[c(\varepsilon)\left(\int_0^{\xi_0}\xi^n\,d\xi + \int_{\xi_0}^{\eta_0}\xi_0^n\,d\eta\right) + M_\alpha\xi_0^n\right]$$

$$\leq \frac{M_g}{n!}[c(\varepsilon)+M_\alpha]^{n-1}\left[\frac{c(\varepsilon)}{n+1} + M_\alpha\right].$$

So, for the derivative $U_{\xi_0}(\xi_0, \eta_0)$ we get the estimation:

$$(3.11) \qquad |U_{\xi_0}(\xi_0, \eta_0)| = |\lim U_{\xi_0}^{(n+1)}(\xi_0, \eta_0)| = |\sum_{k=0}^{\infty}(U_{\xi_0}^{(k+1)} - U_{\xi_0}^{(k)})(\xi_0, \eta_0)|$$

$$\leq M_g \sum_{k=0}^{\infty} \frac{[c(\varepsilon) + M_\alpha]^{k-1}}{k!} \left[\frac{c(\varepsilon)}{k+1} + M_\alpha\right] \leq M_g[c(\varepsilon) + M_\alpha]^{-1} \exp[c(\varepsilon) + 2M_\alpha].$$

Using (3.8) and (3.10), we find

$$|(U_{\eta_0}^{(n+1)} - U_{\eta_0}^{(n)})(\xi_0, \eta_0)| = |-\int_0^{\xi_0} c(\xi, \eta_0)(U^{(n)} - U^{(n-1)})(\xi, \eta_0)]\, d\xi|$$

$$\leq \frac{c(\varepsilon) M_g}{n!}[c(\varepsilon) + M_\alpha]^{n-1} \int_0^{\xi_0} \xi^n\, d\xi \leq \frac{M_g}{(n+1)!}[c(\varepsilon) + M_\alpha]^n \xi_0^{n+1}.$$

Therefore, $U \in C^1(\bar{D}_\varepsilon^{(1)})$ and

$$(3.12) \qquad |U_{\eta_0}(\xi_0, \eta_0)| \leq \xi_0[c(\varepsilon) + M_\alpha]^{-1} \exp[c(\varepsilon) + M_\alpha].$$

Also, by (3.10), it follows that

$$U_{\eta_0\xi_0}^{(n+1)}(\xi_0, \eta_0) = g(\xi_0, \eta_0) - c(\xi_0, \eta_0) U^{(n)}(\xi_0, \eta_0).$$

Thus, the function $U(\xi_0, \eta_0)$ is a solution of (3.3) and $U_{\xi\eta} \in C(\bar{D}_\varepsilon^{(1)})$. Finally, using (3.9) and (3.10), we see that

$$\lim_{n\to\infty} [U_{\eta_0}^{(n+1)} - U_{\xi_0}^{(n+1)} + \alpha(1 - \xi_0) U^{(n+1)}](\xi_0, \eta_0)$$

$$= \alpha(1 - \xi_0) \lim_{n\to\infty} [(U^{(n+1)} - U^{(n)})(\xi_0, \xi_0)] = 0,$$

i.e. $U(\xi_0, \eta_0)$ satisfies boundary conditions (2.18). ■

The next result is very important for the investigation of the singularity of a generalized solution of problem P_α.

LEMMA 3.1. *Let* $c(\xi, \eta)$, $g(\xi, \eta) \in C(\bar{D}_\varepsilon^{(1)})$ *and*

$$(3.13) \qquad g(\xi, \eta) \geq 0, \quad c(\xi, \eta) \leq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)};\ \alpha(\xi) \geq 0 \quad \text{for } 0 \leq \xi \leq 1.$$

Then for the solution $U(\xi, \eta)$ *of the problem* (3.3), (2.18) *(already found in Theorem 3.1) we have*

$$(3.14) \qquad U(\xi, \eta) \geq 0, \quad U_\eta(\xi, \eta) \geq 0, \quad U_\xi(\xi, \eta) \geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}.$$

Proof. In view of (3.7), from (3.13) we have

$$U^{(1)}(\xi_0, \eta_0) = \int_0^{\xi_0} \int_{\xi_0}^{\eta_0} g(\xi, \eta)\, d\eta\, d\xi + 2\int_0^{\xi_0} \int_0^{\eta} g(\xi, \eta)\, d\xi\, d\eta \geq 0.$$

Suppose that $(U^{(n)} - U^{(n-1)})(\xi_0, \eta_0) \geq 0$ for some $n \in \mathbb{N}$. Then

$$(U^{(n+1)} - U^{(n)})(\xi_0, \eta_0) = -\int_0^{\xi_0}\int_{\xi_0}^{\eta_0} c(\xi,\eta)(U^{(n)} - U^{(n-1)})(\xi,\eta)\, d\eta\, d\xi$$

$$-2\int_0^{\xi_0}\int_0^{\eta} c(\xi,\eta)(U^{(n)} - U^{(n-1)})(\xi,\eta)\, d\xi\, d\eta$$

$$+\int_0^{\xi_0} \alpha(1-\xi)(U^{(n)} - U^{(n-1)})(\xi,\xi)\, d\xi \geq 0$$

and

$$U(\xi_0,\eta_0) = \sum_{n=0}^{\infty}(U^{(n+1)} - U^{(n)})(\xi_0,\eta_0) \geq 0. \tag{3.15}$$

Since $U(\xi_0, \eta_0) \geq 0$ for any $(\xi_0, \eta_0) \in \bar{D}_\varepsilon^{(1)}$ and

$$U_{\xi_0}(\xi_0,\eta_0) = \alpha(1-\xi_0)U(\xi_0,\xi_0) \tag{3.16}$$

$$+\int_0^{\xi_0} [g(\xi,\xi_0) - c(\xi,\xi_0)U(\xi,\xi_0)]\, d\xi + \int_{\xi_0}^{\eta_0} [g(\xi_0,\eta) - c(\xi_0,\eta)U(\xi_0,\eta)]\, d\eta,$$

$$U_{\eta_0}(\xi_0,\eta_0) = \int_0^{\xi_0} [g(\xi,\eta_0) - c(\xi,\eta_0)U(\xi,\eta_0)]\, d\xi, \tag{3.17}$$

we conclude that $U_{\xi_0} \geq 0$ and $U_{\eta_0} \geq 0$ in $\bar{D}_\varepsilon^{(1)}$. ∎

As an immediate consequence of Theorem 3.1, (3.16) and (3.17), we have the following

THEOREM 3.2. *Let $c(\xi,\eta) \in C^k(\bar{D}_\varepsilon^{(1)})$, $g(\xi,\eta) \in C^k(\bar{D}_\varepsilon^{(1)})$, $\alpha \in C^k((0,1])$, where $k \geq 1, \varepsilon > 0$. Then there exists a classical solution $U \in C^{k+1}(\bar{D}_\varepsilon^{(1)})$ of the problem $P_{\alpha,2}$.*

4. Existence and uniqueness theorems for $2-D$ Problem $\mathbf{P}_{\alpha,1}$

Consider the problem

$$P_{\alpha,1}: \quad \begin{cases} Lu^{(1)} = \frac{1}{\rho}(\rho u_\rho^{(1)}) - u_{tt}^{(1)} + d(\rho,t)u^{(1)} = f^{(1)} \text{ in } G_\varepsilon, \\ u^{(1)}|_{S_1} = 0, \qquad [u_t^{(1)} + \alpha(\rho)u^{(1)}]|_{S_0} = 0. \end{cases} \tag{4.1}$$

Note that, the notion of the generalized solution of the problem $P_{\alpha,1}$ in the domain G_ε , $\varepsilon \in (0,1)$, has been defined by Definition 2.2.

THEOREM 4.1. *If $d(\rho,t), f^{(1)}(\varrho,t) \in C^1(\bar{G}_0 \setminus (0,0))$, then there exists a generalized solution $u^{(1)} \in C^2(\bar{G}_0 \setminus (0,0))$ of problem $P_{\alpha,1}$ in G_0, which is a classical solution of the problem $P_{\alpha,1}$ in any domain G_ε, $\varepsilon \in (0,1)$.*

Proof. In view of (2.13) and (2.15), i.e. $u^{(2)}(\varrho,t) = \varrho^{1/2}u^{(1)}(\varrho,t)$ and $\xi = 1-\varrho-t$, $\eta = 1-\varrho+t$, consider the function

$$U(\xi,\eta)=u^{(2)}(\varrho(\xi,\eta),t(\xi,\eta)).$$

Then Problem $P_{\alpha,1}$ (see (4.1)) becomes $P_{\alpha,2}$, i.e.

$$U_{\xi\eta} + \frac{1}{4}\left[d^{(2)}(\xi,\eta) + (2-\xi-\eta)^{-2}\right]U = \frac{1}{4\sqrt{2}}(2-\eta-\xi)^{1/2}F(\xi,\eta), \tag{4.2}$$

$$U(0,\eta) = 0, \qquad (U_\eta - U_\xi)(\xi,\xi) + \alpha(1-\xi)U(\xi,\xi) = 0. \tag{4.3}$$

For each $\varepsilon \in (0,1)$ Theorem 3.2 ensures the existence of a classical solution $U(\xi,\eta) \in C^2(\bar{D}_\varepsilon^{(1)})$ of the problem $P_{\alpha,2}$. The inverse transformations lead to a function $u^{(1)}(\varrho,t) \in C^2(\bar{G}_0 \setminus (0,0))$, which is a classical solution of Problem $P_{\alpha,1}$ in G_ε. This solution is also a generalized solution of the same problem in G_0, because each one of test functions $v \in V_0$ is zero in $G_0 \backslash G_\varepsilon$ for some $\varepsilon > 0$ and, for the concrete v, (1.6) coincides with (2.4).

The proof of the theorem is complete. ■

THEOREM 4.2. *For each fixed $\varepsilon \in (0,1)$ there exists at most one generalized solution of the problem $P_{\alpha,1}$ in G_ε.*

Proof. If u_1 and u_2 are two generalized solutions of $P_{\alpha,1}$, then for $u^{(1)} := u_1 - u_2$ we see that

$$u^{(1)} \in C^1(\bar{G}_\varepsilon), \quad u^{(1)}|_{S_1 \cap \bar{G}_\varepsilon} = 0, \quad [u_t^{(1)} + \alpha(r)u^{(1)}]|_{S_0 \cap \bar{G}_\varepsilon} = 0$$

and the identity

$$\int_{G_\varepsilon} [u_t^{(1)} v_t - u_\varrho^{(1)} v_\varrho + d(\varrho,t)u^{(1)}v]\varrho d\varrho dt - \int_{S_0 \cap \partial G_\varepsilon} \varrho\alpha(\varrho)u^{(1)}v \, d\varrho = 0 \tag{4.4}$$

holds for all functions $v \in V_\varepsilon^{(1)}$.

Let $h(\rho,t) \in C^1(\bar{G}_0 \setminus (0,0))$. Set

$$\begin{aligned} g(\xi,\eta) &:= \tfrac{1}{4\sqrt{2}}[2-\xi-\eta]^{1/2}h((2-\xi-\eta)/2, (\eta-\xi)/2) \in C^1(\bar{D}_\varepsilon^{(1)}), \\ c(\xi,\eta) &= \tfrac{1}{4}\left[d(\rho(\xi,\eta), t(\xi,\eta)) + (2-\eta-\xi)^{-2}\right] \in C^1(\bar{D}_\varepsilon^{(1)}), \end{aligned} \tag{4.5}$$

and consider the boundary value problem

$$V_{\xi\eta} + c(\xi,\eta)V = g(\xi,\eta) \quad \text{in } D_\varepsilon, \tag{4.6}$$

$$V\big|_{\eta=1-\varepsilon} = 0, \quad [V_\eta - V_\xi + \alpha(1-\xi)V]\big|_{\eta=\xi} = 0. \tag{4.7}$$

By using the substitutions $\xi_1 = 1-\varepsilon-\eta$, $\eta_1 = 1-\varepsilon-\xi$, and by setting

$$V^{(1)}(\xi_1,\eta_1) = V(1-\varepsilon-\eta_1, 1-\varepsilon-\xi_1), \tag{4.8}$$

the problem (4.6), (4.7) becomes

$$V^{(1)}_{\xi_1\eta_1} + c^{(1)}(\xi_1,\eta_1)V^{(1)} = g^{(1)}(\xi_1,\eta_1) \quad \text{in } D_\varepsilon, \tag{4.9}$$

$$V^{(1)}\big|_{\xi_1=0} = 0, \quad [V^{(1)}_{\eta_1} - V^{(1)}_{\xi_1} + \alpha(\varepsilon+\xi_1)V^{(1)}]\big|_{\eta_1=\xi_1} = 0 \tag{4.10}$$

where

$$c^{(1)}(\xi_1,\eta_1) = \frac{1}{4}\left[d^{(1)}(\xi_1,\eta_1) + (\xi_1+\eta_1+2\varepsilon)^{-2}\right] \in C^1(\bar{D}_\varepsilon).$$

But (4.9), (4.10) is the Goursat–Darboux problem $P_{\alpha,2}$ in the domain D_ε, for which Theorem 3.2 holds. Consequently, there exists a classical solution $V^{(1)}(\xi_1,\eta_1) \in$

C^2 of (4.9), (4.10). The inverse transformation leads to a classical solution $V = V(\xi,\eta)$ of (4.6), (4.7) in D_ε. Similar arguments show that $v(\varrho,t) = \varrho^{-1/2}V(\xi(\varrho,t),\eta(\varrho,t))$ is a classical solution of the problem

$$Lv = \frac{1}{\varrho}(\varrho v_\varrho)_\varrho - v_{tt} + dv = h(\varrho,t) \quad \text{in } G_\varepsilon, \tag{4.11}$$

$$v|_{S_{2,\varepsilon}} = 0, \qquad [v_t + \alpha(\varrho)v]|_{S_0} = 0, \tag{4.12}$$

for fixed $\varepsilon \in (0,1)$.

Multiplying (4.11) by a generalized solution $u^{(1)} \in C^1(\bar{G}_\varepsilon)$ and integrating by parts, we find

$$\int_{G_\varepsilon} [v_t u_t^{(1)} - v_\varrho u_\varrho^{(1)} + dvu^{(1)} - hu^{(1)}]\varrho\, d\varrho\, dt - \int_{S_0\cap\partial G_\varepsilon} \varrho\alpha(\varrho)vu^{(1)}\, d\varrho = 0. \tag{4.13}$$

Comparing (4.13) and (4.4), we see that

$$\int_{G_\varepsilon} h(\varrho,t)u^{(1)}(\varrho,t)\, \rho d\varrho\, dt = 0. \tag{4.14}$$

But the function $h(\varrho,t) \in C^1(\bar{G}_0 \setminus (0,0))$ has been arbitrarily chosen. Thus (4.14) gives $u^{(1)}(\varrho,t) = 0$ in G_ε. The proof is complete. ■

5. Existence and uniqueness theorems for $3-D$ Problem P_α

In this section we consider for the wave equation

$$\Box u := \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} = f(\varrho,\varphi,t), \tag{5.1}$$

subject to the following boundary value problem

$$P_\alpha: \quad \Box u = f \text{ in } \Omega_\varepsilon, \quad u|_{\Sigma_1\cap\partial\Omega_\varepsilon} = 0, \quad [u_t + \alpha(\varrho)u]|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0. \tag{5.2}$$

and prove the following results.

THEOREM 5.1. *For $0 \le \varepsilon < 1$ there exists at most one generalized solution of Problem P_α in Ω_ε.*

Proof. *Case* $0 < \varepsilon < 1$. If u_1, u_2 are two generalized solutions of P_α in Ω_ε, then for $u^{(1)} := u_1 - u_2 \in C^1(\bar{\Omega}_\varepsilon)$ we know that

$$u^{(1)}|_{\Sigma_1\cap\partial\Omega_\varepsilon} = 0, \qquad [u_t^{(1)} + \alpha(\varrho)u^{(1)}]|_{\Sigma_0\cap\partial\Omega_\varepsilon} = 0;$$

and the identity

$$\int\limits_{\Omega_\varepsilon} \left[u_t^{(1)}v_t - u_\rho^{(1)}v_\rho - \frac{1}{\rho^2}u_\varphi^{(1)}v_\varphi\right] \rho\, d\rho\, d\varphi\, dt = \int\limits_{\Sigma_0\cap\partial\Omega_\varepsilon} \rho\alpha(\rho)u^{(1)}v\, d\rho\, d\varphi \tag{5.3}$$

holds for all $v \in V_\varepsilon$. We will show that the Fourier expansion

$$u^{(1)}(\rho,\varphi,t) = \sum_{n=0}^{\infty} \left\{u_n^{(11)}(\rho,t)\cos n\varphi + u_n^{(12)}(\rho,t)\sin n\varphi\right\} \tag{5.4}$$

has zero Fourier–coefficients $u_n^{(1i)}(\varrho,t)$ in Ω_ε, i.e. $u^{(1)} \equiv 0$ in Ω_ε.

Since $u^{(1)} \in C^1(\bar{\Omega}_\varepsilon)$, using

$$v_1(\rho,\varphi,t) = w(\rho,t)\cos n\varphi \in V_\varepsilon \text{ or } v_2(\rho,\varphi,t) = w(\rho,t)\sin n\varphi \in V_\varepsilon$$

in (5.3), we derive

$$\int_{G_\varepsilon}\left[u_{n,t}^{(1i)}w_t - u_{n,\rho}^{(1i)}w_\rho - \frac{n^2}{\rho^2}u_n^{(1i)}w\right]\rho\, d\rho\, dt - \int_{\partial G_\varepsilon \cap S_0}\rho\alpha(\rho)u_n^{(1i)}w\, d\rho = 0 \tag{5.5}$$

for all $w \in V_\varepsilon^{(1)}, n \in \mathbb{N}, i = 1,2$. From Definition 2.2 it follows that the functions $u_n^{(1i)}(\varrho,t)$ are generalized solutions of the homogeneous problem $P_{\alpha,1}$ with $d(\varrho,t) = n^2\rho^{-2} \in C^\infty(\bar{G}_0\backslash(0,0))$. Clearly Theorem 4.2 gives $u_n^{(1i)}(\varrho,t) \equiv 0$ in Ω_ε for $n \in \mathbb{N}$, $i = 1,2$ and thus $u^{(1)} = u_1 - u_2 \equiv 0$ in Ω_ε.

Case $\varepsilon = 0$. In this case from Lemma 2.1 it follows that the generalized solution $u^{(1)} \in C^1(\bar{\Omega}_0 \setminus (0,0,0))$ of Problem P_α in Ω_0 is also a generalized solution of the homogeneous problem P_α in Ω_ε for each $\varepsilon \in (0,1)$. From Case 1 we know that $u^{(1)} \equiv 0$ in Ω_ε for each $\varepsilon > 0$ and thus $u^{(1)} = u_1 - u_2 \equiv 0$ in Ω_0. ∎

THEOREM 5.2. *Let the function* $f \in C(\bar{\Omega}_0) \cap C^1(\bar{\Omega}_0\backslash(0,0,0))$ *be of the form:*

$$f^{(1)}(\varrho,\varphi,t) = \sum_{n=0}^{k}\left\{f_n^{(11)}(\varrho,t)\cos n\varphi + f_n^{(12)}(\varrho,t)\sin n\varphi\right\}. \tag{5.6}$$

Then there exists one and only one generalized solution

$$u^{(1)}(\varrho,\varphi,t) = \sum_{n=0}^{k}\left\{u_n^{(11)}(\varrho,t)\cos n\varphi + u_n^{(12)}(\varrho,t)\sin n\varphi\right\} \tag{5.7}$$

of the problem P_α *in* Ω_0, $u^{(1)} \in C^2(\bar{\Omega}_0 \setminus (0,.,0))$ *and it is a classical solution of the problem* P_α *in each domain* $\Omega_\varepsilon, \varepsilon \in (0,1)$. *Moreover, for a fixed* n *the corresponding trigonometric polynomial* u_n *of degree* n *satisfies a priori estimates: for* $n = 0$:

$$\begin{aligned} \|u_0(x_1,x_2,t)\|_{C^1(\tilde{\Omega}_\varepsilon)} &= \sum_{|\alpha|\le 1}\sup_{\tilde{\Omega}_\varepsilon}|D^\alpha u_0|\cdot \\ &\le 8\exp(2M_\alpha)\varepsilon^{1/2}\exp(1/4\varepsilon^2)\|f_0^{(11)}\|_{C^0(\bar{G}_0)}; \end{aligned} \tag{5.8}$$

for $n \in \mathbb{N}$:

$$\begin{aligned} &\|u_n(x_1,x_2,t)\|_{C^1(\tilde{\Omega}_\varepsilon)} \\ &\le 8\exp(2M_\alpha)\frac{\varepsilon^{1/2}}{n}\exp\left(\frac{n^2}{\varepsilon^2}\right)\left(\|f_n^{(11)}\|_{C^0(\bar{G}_0)} + \|f_n^{(12)}\|_{C^0(\bar{G}_0)}\right), \end{aligned} \tag{5.9}$$

where $\tilde{\Omega}_\varepsilon = \Omega_0 \cap \{(\varrho,t) : \varrho + t > \varepsilon\}$.

Proof. It is enough to consider the case of a fixed number n. Let

$$U^{(1)}(\varrho,t) = \begin{cases} u_n^{(11)}(\varrho,t) \text{ in case } F^{(1)}(\varrho,t) = f_n^{(11)}(\varrho,t), \\ u_n^{(12)}(\varrho,t) \text{ in case } F^{(1)}(\varrho,t) = f_n^{(12)}(\varrho,t). \end{cases} \tag{5.10}$$

Then by (5.7) and (5.10), the equation (5.1) becomes

$$\frac{1}{\varrho}(\varrho U_\varrho^{(1)})_\varrho - U_{tt}^{(1)} - \frac{n^2}{\varrho^2}U^{(1)} = F^{(1)}(\varrho,t) \tag{5.11}$$

As in Section 2, we make the substitutions

$$\xi = 1 - \varrho - t\ ,\ \eta = 1 - \varrho + t, \tag{5.12}$$

and introduce the new function

$$U^{(2)}(\xi,\eta) = \varrho^{1/2} U^{(1)}(\varrho(\xi,\eta), t(\xi,\eta)). \tag{5.13}$$

Then (5.11) reduces to (2.18), where

$$\begin{aligned} c(\xi,\eta) &= \frac{1-4n^2}{4(2-\eta-\xi)^2} \in C^\infty(\bar{D}_0 \setminus (1,1))\ , \\ g(\xi,\eta) &= \frac{1}{4\sqrt{2}}(2-\eta-\xi)^{1/2} f_n^{(2i)}(\xi,\eta) \in C^1(\bar{D}_0 \backslash (1,1))\ , \\ f_n^{(2i)}(\xi,\eta) &= f_n^{(1i)}(\varrho(\xi,\eta), t(\xi,\eta)), \end{aligned} \tag{5.14}$$

and satisfies the Goursat–Darboux problem $P_{\alpha,2}$. Theorems 3.1 and 3.2 ensure the existence of a classical solution $U^{(2)} = U^{(2)}(\xi,\eta)$ of this problem with the properties (3.6).

Case $n \in \mathbb{N}$. In view of (3.5), (5.14), it is easy to see that

$$\begin{aligned} c(\varepsilon) &:= \sup_{D_\varepsilon^{(1)}} |c(\xi,\eta)| \le \frac{n^2}{\varepsilon^2}\ , \\ M_g &:= \sup_{D_\varepsilon^{(1)}} |\frac{1}{4\sqrt{2}}(2-\eta-\xi)^{1/2} f_n^{(2i)}(\xi,\eta)| \le \frac{1}{4} \|f_n^{(1i)}\|_{C^0(\bar{G}_0)}\ . \end{aligned} \tag{5.15}$$

where $D_\varepsilon^{(1)} = \{(\xi,\eta) | 0 < \xi < \eta < 1,\ 0 < \xi < 1-\varepsilon\},\ \varepsilon > 0$. Hence Theorems 3.1 and 3.2, on one hand, ensure the smoothness of the solution $U^{(2)}$ of Problem $P_{\alpha,2}$, i.e.

$$U_n^{(2i)}(\xi,\eta) := U^{(2)} \in C^2(\bar{D}_\varepsilon^{(1)}), \tag{5.16}$$

on the other hand, they ensure the a priori estimates:

$$\begin{aligned} \sup_{D_\varepsilon^{(1)}} |U_n^{(2i)}(\xi,\eta)| &\le \frac{1}{4} \|f_n^{(1i)}\|_{C^\circ(\bar{G}_0)} \frac{\varepsilon^2}{n^2} \exp(M_\alpha) \exp\left(\frac{n^2}{\varepsilon^2}\right)\ , \\ \sup_{D_\varepsilon^{(1)}} \{|U_{n,\xi}^{(2i)}|,\ |U_{n,\eta}^{(2i)}|\} &\le \frac{1}{4} \|f_n^{(1i)}\|_{C^\circ(\bar{G}_0)} \frac{\varepsilon^2}{n^2} \exp(2M_\alpha) \exp\left(\frac{n^2}{\varepsilon^2}\right)\ . \end{aligned} \tag{5.17}$$

Also, by (5.12) and (5.13), we have

$$U_n^{(1i)}(\varrho,t) = \varrho^{-\frac{1}{2}} U_n^{(2i)}(\xi,\eta)\ .$$

Since $\varrho \ge \varepsilon/2$ for $(\xi,\eta) \in D_\varepsilon^{(1)}$, by the inverse transformation:

$$\begin{aligned} |u_n^{(1i)}(\varrho,t)| &\le \exp(M_\alpha) \frac{\varepsilon^{3/2}}{n^2} \exp\left(\frac{n^2}{\varepsilon^2}\right) \|f_n^{(1i)}\|_{C^\circ(\bar{G}_0)}\ , \\ |u_{n,t}^{(1i)}(\varrho,t)| &\le \exp(2M_\alpha) \frac{\varepsilon^{3/2}}{n^2} \exp\left(\frac{n^2}{\varepsilon^2}\right) \|f_n^{(1i)}\|_{C^\circ(\bar{G}_0)}\ , \\ |u_{n,\varrho}^{(1i)}(\varrho,t)| &\le 2\exp(2M_\alpha) \frac{\varepsilon^{1/2}}{n^2} \exp\left(\frac{n^2}{\varepsilon^2}\right) \|f_n^{(1i)}\|_{C^\circ(\bar{G}_0)}\ . \end{aligned} \tag{5.18}$$

Therefore, in view of (5.7) and (5.18), we derive

$$\begin{aligned}(5.19)\qquad &\|\frac{1}{\varrho}u^{(1)}_{n,\varphi}(\varrho,\varphi,t)\|_{C^\circ(\tilde{\Omega}_\varepsilon)}\\ &\le \exp(2M_\alpha)\frac{\varepsilon^{1/2}}{n}\exp\left(\frac{n^2}{\varepsilon^2}\right)\left(\|f^{(11)}_n\|_{C^\circ(\bar{G}_0)}+\|f^{(12)}_n\|_{C^\circ(\bar{G}_0)}\right).\end{aligned}$$

Since $u_n(\varrho\cos\varphi,\varrho\sin\varphi,t)=u^{(1)}_n(\varrho,\varphi,t)$, obviously

$$|u_{n,x_i}(x_1,x_2,t)|\le 3\exp(2M_\alpha)\frac{\varepsilon^{1/2}}{n}\exp\left(\frac{n^2}{\varepsilon^2}\right)\left(\|f^{(11)}_n\|_{C^\circ(\bar{G}_0)}+\|f^{(12)}_n\|_{C^\circ(\bar{G}_0)}\right),$$

$i=1,2$. So, the estimate (5.9) holds in $\tilde{\Omega}_\varepsilon$.

Case $n=0$. In this case, by (5.6) and (5.7), it follows that $f^{(1)}_0(\varrho,\varphi,t)=f^{(11)}_0(\varrho,t)$ and $u_0(x_1,x_2,t)=u^{(1)}_0(\varrho,\varphi,t)=u^{(11)}_0(\varrho,t)$. Problem $P_{\alpha,2}$ in this case becomes

$$U^{(2)}_{\xi\eta}+c(\xi,\eta)U^{(2)}=g(\xi,\eta)\ ,\ U^{(2)}|_{\xi=0}=0\ ,\ U^{(2)}|_{\eta=\xi}=0,$$

where

$$c(\xi,\eta)=[2(2-\eta-\xi)]^{-2}\in C^\infty(\bar{D}_0\setminus\{1,1\})$$

and

$$c(\varepsilon)=\sup_{D^{(1)}_\varepsilon}|c(\xi,\eta)|\le\frac{1}{4\varepsilon^2}\ ,\ M_g\le\frac{1}{4}\|f^{(11)}_0\|_{C^\circ(\bar{G}_0)}$$

Arguments similar to the previous case lead to (5.8).■

The following theorem is an immediate consequence of Theorems 5.1 and 5.2

THEOREM 5.3. *Let the function $f\in C^1(\bar{\Omega}_0)$ be of the form*

$$(5.20)\qquad f(\rho,\varphi,t)=\sum_{n=0}^{\infty}\{f^{(1)}_n(\rho,t)\cos n\varphi+f^{(2)}_n(\rho,t)\sin n\varphi\}.$$

Suppose that the Fourier coefficients $f^{(1)}_n(\rho,t)$ and $f^{(2)}_n(\rho,t)$ satisfy

$$\begin{aligned}(5.21)\qquad &\|f\|_{\exp(\varepsilon)}:=\exp\left(\frac{1}{4\varepsilon^2}\right)\|f^{(11)}_0\|_{C^\circ(\bar{G}_0)}\\ &+\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(\frac{n^2}{\varepsilon^2}\right)\left(\|f^{(11)}_n\|_{C^\circ(\bar{G}_0)}+\|f^{(12)}_n\|_{C^\circ(\bar{G}_0)}\right)<\infty.\end{aligned}$$

Then there exist one and only one generalized solution $u\in C^1(\tilde{\Omega}_\varepsilon)$ of the problem P_α in Ω_ε and the a priori estimate

$$(5.22)\qquad \|u\|_{C^1(\tilde{\Omega}_\varepsilon)}\le 8\exp(2M_\alpha)\|f\|_{\exp(\varepsilon)}$$

holds. If the series (5.20) is finite, then $u\in C^2(\bar{\Omega}_0\setminus(0,0,0))$ and it is a classical solution of the problem P_α in $\Omega_\varepsilon,\varepsilon\in(0,1)$

REMARK 5.1. *Condition* (5.21) *is valid for each* $\varepsilon \in (0,1)$ *if there exists a function* ψ *with* $\psi(n) \to \infty$ *as* $n \to \infty$, *such that*

$$\sum_{n=1}^{\infty} \frac{1}{n} \exp(n^2\psi(n)) \left(\|f_n^{(11)}\|_{C^0(\bar{G}_0)} + \|f_n^{(12)}\|_{C^0(\bar{G}_0)} \right) < \infty. \tag{5.23}$$

REMARK 5.2. *As we see, the norm* (5.21) *on the right-hand side of* (5.22) *tends to infinity as* $\varepsilon \to 0$. *At this point, it is reasonable to remained that, according to Theorem 6.1 (see, the discussion in Introduction) the estimate* (5.22) *is satisfied also by the generalized solutions which have singularities at the point* $(0,0,0)$. *Therefore, the left-hand side of* (5.22) *tends to infinity as* $\rho \to 0$. *The above phenomenon is subject to the new paper* [9].

6. On the singularity of solutions of Problem P_α

For the wave equation

$$\Box u = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2} u_{\varphi\varphi} - u_{tt} = f(\varrho, \varphi, t) \tag{6.1}$$

we consider again the boundary value problem P_α, i.e.

$$P_\alpha : \qquad \Box u = f \text{ in } \Omega_0, \qquad u|_{\Sigma_1} = 0, \qquad [u_t + \alpha(\varrho)u]|_{\Sigma_0} = 0. \tag{6.2}$$

and begin with the following interesting result of this section

THEOREM 6.1. *Let* $\alpha(\varrho) \geq 0$, $\varrho \in [0,1]$; $\alpha(\varrho) \in C^\infty([0,1])$. *Then for each* $n \in \mathbb{N}$, $n \geq 4$, *there exists a function* $f_n(\varrho, \varphi, t) \in C^{n-2}(\bar{\Omega}_0)$, *for which the corresponding general solution* u_n *of the problem* P_α *belongs to* $C^n(\bar{\Omega}_0 \backslash (0,0,0))$ *and the estimate*

$$|u_n(\rho, \varphi, \rho)| \geq \frac{1}{2}|u_n(2\rho, \varphi, 0)| + \rho^{-n}|\cos n\varphi| \geq \rho^{-n}|\cos n\varphi|, \qquad 0 < \rho < 1, \tag{6.3}$$

holds. In the case $\alpha(\varrho) \equiv 0$ *the upper estimate*

$$|u_n(\varrho, \varphi, t)| \leq c_\mu \rho^{-1/2} \left(\frac{\rho}{(\rho + t)(\rho - t)} \right)^{n - \frac{1}{2}} |\cos n\varphi|, (\varrho, t) \in D_1^\mu \tag{6.4}$$

holds, where $c_\mu = const$ *and*

$$D_1^\mu := \{(\rho, t) : 0 < \rho - t \leq \rho + t \leq \mu(\rho - t)\}, \mu < 2^{\frac{2n+1}{2n-1}} - 1.$$

Thus, for $\alpha(\varrho) \equiv 0$ *we have two-sided estimates, which in special cases* $t = \rho$ *and* $t = 0$ *are:*

$$\rho^{-n}|\cos n\varphi| \leq |u_n(\rho, \varphi, \rho)|, \qquad |u_n(\rho, \varphi, 0)| \leq C_2\rho^{-n}|\cos n\varphi|, \tag{6.5}$$

with $C_2 = const$. *That is, in the case of Problem P2 the exact behavior of* $u_n(x_1, x_2, t)$ *around (0,0,0) is* $(x_1^2 + x_2^2)^{-n/2}$.

Proof. Note that, by Theorem 1.1, the functions

$$w_n(\varrho, \varphi, t) = \varrho^{-n}(\varrho^2 - t^2)^{n-1/2}(a_n \cos n\varphi + b_n \sin n\varphi), n \geq 4,$$

are classical solutions of Problem P_α^* with $\alpha \equiv 0$,.where obviously $w_n \in C^{n-2}(\bar{\Omega}_0)$.

We consider the special case of Problem P_α :

$$\Box u = \varrho^{-n}(\varrho^2 - t^2)^{n-1/2} \cos n\varphi \qquad \text{in } \Omega_0, \tag{6.6}$$

$$u|_{\Sigma_1} = 0, \qquad [u_t + \alpha(\varrho)u]|_{\Sigma_0} = 0. \tag{6.7}$$

The Theorem 5.1 declares that the problem (6.6), (6.7) has at most one generalized solution. On the other hand, from Theorem 5.2 we know that for this right-hand side there exists a generalized solution in Ω_0 of the form

$$u_n(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi \in C^{n-1}(\bar{\Omega}_0\backslash(0,0,0)),$$

which is classical solution in Ω_ε, $\varepsilon \in (0,1)$. By setting $u_n^{(2)}(\varrho,t) = \varrho^{\frac{1}{2}}u_n^{(1)}(\varrho,t)$ and substituting

$$\xi = 1-\varrho-t, \qquad \eta = 1-\varrho+t, \tag{6.8}$$

the problem (6.6), (6.7), in view of

$$U_n(\xi,\eta) = u_n^{(2)}(\varrho(\xi,\eta),t(\xi,\eta)), \tag{6.9}$$

becomes a Goursat–Darboux problem $P_{\alpha,2}$:

$$U_{n,\xi\eta} + c(\xi,\eta)U_n = g(\xi,\eta), \tag{6.10}$$

$$U_n(0,\eta) = 0, \qquad [U_{n,\eta} - U_{n,\xi} + \alpha(1-\xi)U_n]\big|_{\eta=\xi} = 0. \tag{6.11}$$

The coefficients

$$c(\xi,\eta) = \frac{1-4n^2}{4(2-\eta-\xi)^2} \in C^\infty(\bar{D}_\varepsilon^{(1)}), \quad n \geq 4, \tag{6.12}$$

$$g(\xi,\eta) = 2^{n-\frac{5}{2}}\left[\frac{(1-\xi)(1-\eta)}{2-\eta-\xi}\right]^{n-\frac{1}{2}} \in C^{n-1}(\bar{D}_\varepsilon^{(1)}) \tag{6.13}$$

are defined by (3.1). It is obvious that in this case $c(\xi,\eta) \leq 0$, $g(\xi,\eta) \geq 0$ in $\bar{D}_\varepsilon^{(1)}$, $\varepsilon \in (0,1)$.

Thus, for $\alpha(\xi) \geq 0$, in view of Theorem 3.1 and Lemma 3.1,we have the following result.

Proposition 6.1. *There exists a classical solution $U(\xi,\eta) \in C^n(\bar{D}_0 \setminus (1,1))$ for the problem* (6.10), (6.11) *for which*

$$U(\xi,\eta) \geq 0, \quad U_\xi(\xi,\eta) \geq 0, \quad U_\eta(\xi,\eta) \geq 0 \text{ in } \bar{D}_\varepsilon^{(1)}.$$

Let

$$K = \int_{D_{\frac{1}{2}}^{(1)}} g^2(\xi,\eta)\,d\eta\,d\xi > 0. \tag{6.14}$$

From (6.10) for $0 < \varepsilon < 1/2$ it follows that

$$\begin{aligned} 0 < K \leq \int_{D_\varepsilon^{(1)}} g^2(\xi,\eta)\,d\eta\,d\xi &= \int_{D_\varepsilon^{(1)}} U_{\xi\eta}g(\xi,\eta)\,d\eta\,d\xi \\ &+ \int_{D_\varepsilon^{(1)}} c(\xi,\eta)U(\xi,\eta)g(\xi,\eta)\,d\eta\,d\xi =: I_1 + I_2, \end{aligned} \tag{6.15}$$

where

$$I_1 = \int_0^{1-\varepsilon} \int_\xi^1 (U_{\xi\eta} g)(\xi,\eta)\, d\eta\, d\xi$$

$$= \int_0^{1-\varepsilon} [U_\xi(\xi,1)g(\xi,1) - U_\xi(\xi,\xi)g(\xi,\xi)]\, d\xi - \int_{D_\varepsilon^{(1)}} (U_\xi g_\eta)(\xi,\eta)\, d\eta\, d\xi.$$

By (6.13), it is obvious that $g(\xi,1)=0$. So,

$$I_1 = -\int_0^{1-\varepsilon} U_\xi(\xi,\xi)g(\xi,\xi)\, d\xi - \int_{D_\varepsilon^{(1)}} (U_\xi g_\eta)(\xi,\eta)\, d\eta\, d\xi. \tag{6.16}$$

Since

$$\int_{D_\varepsilon^{(1)}} (U_\xi g_\eta)(\xi,\eta)\, d\xi\, d\eta = \int_0^{1-\varepsilon}\int_0^\eta (U_\xi g_\eta)(\xi,\eta)\, d\xi\, d\eta$$

$$+ \int_{1-\varepsilon}^1 \int_0^{1-\varepsilon} (U_\xi g_\eta)(\xi,\eta)\, d\xi\, d\eta = \int_0^{1-\varepsilon} [(Ug_\eta)(\eta,\eta) - (Ug_\eta)(0,\eta)]\, d\eta$$

$$+ \int_{1-\varepsilon}^1 [(Ug_\eta)(1-\varepsilon,\eta) - (Ug_\eta)(0,\eta)]\, d\eta - \int_{D_\varepsilon^{(1)}} Ug_{\xi\eta}(\xi,\eta)\, d\xi\, d\eta \tag{6.17}$$

$$= \int_0^{1-\varepsilon} (Ug_\eta)(\eta,\eta)\, d\eta + \int_{1-\varepsilon}^1 (Ug_\eta)(1-\varepsilon,\eta)\, d\eta$$

$$- \int_{D_\varepsilon^{(1)}} (Ug_{\xi\eta})(\xi,\eta)\, d\xi\, d\eta,$$

(6.16) becomes

$$\begin{aligned} I_1 = &-\int_0^{1-\varepsilon} [U_\xi(\xi,\xi)g(\xi,\xi) + U(\xi,\xi)g_\eta(\xi,\xi)]\, d\xi \\ &- \int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta + \int_{D_\varepsilon^{(1)}} (Ug_{\xi\eta})(\xi,\eta)\, d\xi\, d\eta. \end{aligned} \tag{6.18}$$

An elementary calculation shows that

$$g_\xi(\xi,\eta) = -(n-\frac{1}{2})2^{n-\frac{5}{2}} \left[\frac{(1-\xi)(1-\eta)}{2-\eta-\xi}\right]^{n-\frac{3}{2}} \left[\frac{(1-\eta)}{2-\eta-\xi}\right]^2 \le 0, \tag{6.19}$$

$$g_\eta(\xi,\eta) = -(n-\frac{1}{2})2^{n-\frac{5}{2}} \left[\frac{(1-\xi)(1-\eta)}{2-\eta-\xi}\right]^{n-\frac{3}{2}} \left[\frac{(1-\xi)}{2-\eta-\xi}\right]^2 \le 0, \tag{6.20}$$

and

$$g_\xi(\xi,\xi) = g_\eta(\xi,\xi) = \frac{1}{16}(1-2n)(1-\xi)^{n-\frac{3}{2}}. \tag{6.21}$$

From (6.18) and (6.15) it follows that

$$\begin{aligned} 0 < K \le I_1 + I_2 = &- \int_0^{1-\varepsilon} [U_\xi(\xi,\xi)g(\xi,\xi) + U(\xi,\xi)g_\xi(\xi,\xi)]\, d\xi \\ &- \int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta + \int_{D_\varepsilon^{(1)}} U[g_{\xi\eta} + cg](\xi,\eta)\, d\xi\, d\eta. \end{aligned} \tag{6.22}$$

Also, it is easy to check that

$$g_{\xi\eta}(\xi,\eta) + c(\xi,\eta)g(\xi,\eta) = 0.$$

Thus,

$$\begin{aligned} 0 < K \le I_1 + I_2 = &- \int_0^{1-\varepsilon} [U_\xi(\xi,\xi)g(\xi,\xi) + U(\xi,\xi)g_\xi(\xi,\xi)]\, d\xi \\ &- \int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta, \end{aligned} \tag{6.23}$$

where, as it is easy to check,

$$g_\xi(\xi,\xi) = \frac{1}{2}[g(\xi,\xi)]_\xi. \tag{6.24}$$

The function $U(\xi,\eta)$ is a classical solution of (6.10), (6.11) in $\bar{D}_\varepsilon$, $\varepsilon \in (0,1)$ with

$$U_\xi(\xi,\xi) = \frac{1}{2}[U(\xi,\xi)]_\xi + \frac{1}{2}\alpha(1-\xi)U(\xi,\xi) \tag{6.25}$$

If we substitute (6.24) and (6.25) into (6.23), we get

$$\begin{aligned} K \le I_1 + I_2 &= -\frac{1}{2}\int_0^{1-\varepsilon} [g(\xi,\xi)U(\xi,\xi)]_\xi\, d\xi \\ &-\frac{1}{2}\int_0^{1-\varepsilon} \alpha(1-\xi)U(\xi,\xi)g(\xi,\xi)\, d\xi - \int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta \\ &= -\frac{1}{2}(gU)(1-\varepsilon,1-\varepsilon) - \frac{1}{2}\int_0^{1-\varepsilon} \alpha(1-\xi)U(\xi,\xi)g(\xi,\xi)\, d\xi \\ &- \int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\, d\eta. \end{aligned} \tag{6.26}$$

According to Proposition 6.1 and the choice of right-hand side of (6.8), we have

$$U(\xi,\eta) \ge 0, U_\eta(\xi,\eta) \ge 0, \alpha(\xi) \ge 0, g(\xi,\eta) \ge 0, g_\eta(\xi,\eta) \le 0 \text{ in } \bar{D}_\varepsilon^{(1)},$$

which together with (6.26) implies

$$K \leq I_1 + I_2 \leq -\int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta) g_\eta(1-\varepsilon,\eta)\, d\eta - \frac{1}{2}(gU)(1-\varepsilon, 1-\varepsilon)$$

$$= \int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta) |g_\eta(1-\varepsilon,\eta)|\, d\eta - \frac{1}{2}(gU)(1-\varepsilon, 1-\varepsilon)$$

$$\leq \int_{1-\varepsilon}^{1} U(1-\varepsilon,1) |g_\eta(1-\varepsilon,\eta)|\, d\eta - \frac{1}{2}(gU)(1-\varepsilon, 1-\varepsilon)$$

$$= \left[U(1-\varepsilon,1) - \frac{1}{2}U(1-\varepsilon,1-\varepsilon)\right] g(1-\varepsilon,1-\varepsilon),$$

because $g(1-\varepsilon,1)=0$. Since $g(1-\varepsilon,1-\varepsilon) = \frac{1}{4}\varepsilon^{n-\frac{1}{2}}$, we see that

$$0 < K \leq \left[U(1-\varepsilon,1) - \frac{1}{2}U(1-\varepsilon,1-\varepsilon)\right] \frac{1}{4}\varepsilon^{n-\frac{1}{2}}.$$

For $\xi = 1-\varepsilon$, $\eta = 1$ we have $\varrho = t = \varepsilon/2$ and so

$$0 < 4K\varepsilon^{\frac{1}{2}-n} \leq u_n^{(2)}\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right) - \frac{1}{2}u_n^{(2)}(\varepsilon,0). \tag{6.27}$$

Finally, the inverse transformation gives

$$u_n^{(1)}\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right) \geq \frac{1}{2}u_n^{(1)}(\varepsilon,0) + \tilde{C}_1\varepsilon^{-n} \geq \tilde{C}_1\varepsilon^{-n},\ 0<\varepsilon<\frac{1}{2},$$

with $\tilde{C}_1 = 2^{\frac{5}{2}}K$. Multiplying the function u_n by $\tilde{C}_1^{-1}$, we see that (6.3) holds.

In order to obtain an upper estimate of the singular solution, we consider the case $\alpha(\varrho)\equiv 0$. In this case (6.26) gives

$$I_1 + I_2 = \int_{\bar{D}_\varepsilon^{(1)}} g^2(\xi,\eta)\, d\xi\, d\eta = -\frac{1}{2}(Ug)(1-\varepsilon,1-\varepsilon) - \int_{1-\varepsilon}^{1} (Ug_\eta)(1-\varepsilon,\eta)\, d\eta$$

Put

$$K_1 = \int_{D_0^{(1)}} g^2(\xi,\eta)\, d\xi\, d\eta > 0.$$

Then for $0<\delta<\varepsilon<1$ we have

$$
\begin{aligned}
K_1 &\geq I_1 + I_2 \\
&= -\frac{1}{2}(gU)(1-\varepsilon, 1-\varepsilon) + \int_{1-\varepsilon}^{1} U(1-\varepsilon, \eta)|g_\eta(1-\varepsilon, \eta)|\, d\eta \\
&\geq -\frac{1}{2}(gU)(1-\varepsilon, 1-\varepsilon) + \int_{1-\delta}^{1} U(1-\varepsilon, 1-\varepsilon)|g_\eta(1-\varepsilon, \eta)|\, d\eta \\
&\geq -\frac{1}{2}(gU)(1-\varepsilon, 1-\varepsilon) + \int_{1-\delta}^{1} U(1-\varepsilon, 1-\delta)|g_\eta(1-\varepsilon, \eta)|\, d\eta \\
&\geq -\frac{1}{2}(gU)(1-\varepsilon, 1-\varepsilon) + (gU)(1-\varepsilon, 1-\delta) \\
&\geq U(1-\varepsilon, 1-\delta)\left[g(1-\varepsilon, 1-\delta) - \frac{1}{2}g(1-\varepsilon, 1-\varepsilon)\right] \\
&\geq \lambda(gU)(1-\varepsilon, 1-\delta),
\end{aligned} \tag{6.28}
$$

where the constant $\lambda > 0$ is such that

$$(1-\lambda)g(1-\varepsilon, 1-\delta) \geq g(1-\varepsilon, 1-\varepsilon). \tag{6.29}$$

Using the explicit formula (6.16) for the function $g(\xi, \eta)$, we see that the last inequality is equivalent to

$$(1-\lambda)\left(\frac{\delta}{\varepsilon+\delta}\right)^{n-\frac{1}{2}} \geq 2^{-n+\frac{1}{2}}, \tag{6.30}$$

which implies

$$0 < \lambda \leq 1 - \frac{1}{2}\left(\frac{\varepsilon+\delta}{2\delta}\right)^{n-\frac{1}{2}}. \tag{6.31}$$

A necessary condition, for (6.31) to be satisfied is

$$1 \leq \frac{\varepsilon}{\delta} < 2^{\frac{2n+1}{2n-1}} - 1. \tag{6.32}$$

Using (6.32), we can find an upper estimate for the generalized solution u_n in this concrete case. To do that we consider the domain

$$D^\mu := \{(\xi, \eta) : 1-\eta \leq 1-\xi \leq \mu(1-\eta)\}, \tag{6.33}$$

where $1 \leq \mu < 2^{\frac{2n+1}{2n-1}} - 1$. Observe that

$$\inf_{D^\mu}\left\{1 - \frac{1}{2}\left(\frac{1-\xi+1-\eta}{2(1-\eta)}\right)^{n-\frac{1}{2}}\right\} = 1 - \frac{1}{2}\left(\frac{1+\mu}{2}\right)^{n-\frac{1}{2}} =: C_\mu > 0.$$

For $\lambda = C_\mu$, the inequalities (6.30) and (6.29) are satisfied and so, by (6.28), we see that

$$U(\xi, \eta) \leq 2^{-n+5/2} K_1 C_\mu^{-1} \left(\frac{2-\xi-\eta}{(1-\xi)(1-\eta)}\right)^{n-\frac{1}{2}}, (\xi, \eta) \in D^\mu. \tag{6.34}$$

By (6.9) and (6.8), the inequality (6.34) transforms to

$$u_n^{(2)}(\varrho, t) \leq 4K_1 C_\mu^{-1} \left(\frac{\rho}{(\rho+t)(\rho-t)}\right)^{n-\frac{1}{2}}, \tag{6.35}$$

which is satisfied for

$$(\varrho,t) \in D_1^{\mu} := \{0 < \rho - t \le \rho + t \le \mu(\rho - t)\}.$$

Finally, (6.35) implies

$$u_n^{(1)}(\varrho,t) \le 4K_1 C_\mu^{-1} \rho^{-1/2} \left(\frac{\rho}{(\rho+t)(\rho-t)}\right)^{n-\frac{1}{2}} \quad \text{for } (\varrho,t) \in D_1^{\mu}, \tag{6.36}$$

which coincides with the estimate (6.4)

Note that $C_\mu = 1/2$ on $\{t = 0\}$ and so

$$u_n^{(1)}(\rho,0) \le 8K_1\rho^{-n}, \qquad 0 < \rho < 1, \tag{6.37}$$

which is the upper estimate in (6.5). The proof of theorem is complete. ■

We conclude this section with

THEOREM 6.2. *Let $\alpha(\varrho) \ge 0$ for $\rho \in [0,1]$, $\alpha \in C^{n-2}[0,1]$. Then for $n \in \mathbb{N}$, $n \ge 4$ there exists a function $f_{n1}(\varrho,\varphi,t) \in C^{n-2}(\bar{\Omega}_0)$ (different from the function of Theorem 6.1) such that for the corresponding to it generalized solution u_n of the problem P_α*

$$u_n(\rho,\varphi,t) \in C^{n-1}(\bar{\Omega}_0 \setminus (0,0,0)),$$

$$u_n(\rho,\varphi,\rho) \ge u_n(2\rho,\varphi,0) + \rho^{1-n}|\cos n\varphi| \ge \rho^{1-n}|\cos n\varphi|. \tag{6.38}$$

Proof. The functions

$$v_n(\rho,\varphi,t) = t\rho^{-n}(\rho^2 - t^2)^{n-3/2}(a_n \cos n\varphi + b_n \sin n\varphi)$$

are classical solutions of Protter's problem $P1^*$. We consider the problem

$$\Box u = t\rho^{-n}(\rho^2 - t^2)^{n-3/2}\cos n\varphi \tag{6.39}$$

$$u|_{\Sigma_1} = 0, \; [u_t + \alpha(\rho)u]|_{\Sigma_0} = 0. \tag{6.40}$$

According to Theorem 5.1, the problem (6.39), (6.40) has at most one generalized solution. Simultaneously Theorem 5.2 for this right-hand side ensure the existence of a generalized solution in Ω_0, which is of the form

$$u_n(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi \in C^{n-1}(\bar{\Omega}_0 \backslash (0,0,0))$$

and is a classical solution in Ω_ε, $\varepsilon \in (0,1)$.

Using the substitutions $u_n^{(2)}(\varrho,t) = \varrho^{\frac{1}{2}} u_n^{(1)}(\varrho,t)$, (6.8) and (6.9), the problem (6.39), (6.40) becomes a Goursat-Darboux problem

$$U_{n,\xi\eta} + c(\xi,\eta)U_n = g(\xi,\eta), \tag{6.41}$$

$$U_n(0,\eta) = 0, \qquad [U_{n,\eta} - U_{n,\xi} + \alpha(1-\xi)U_n]|_{\eta=\xi} = 0, \tag{6.42}$$

where $c(\xi,\eta)$ is defined by (6.12), while

$$g(\xi,\eta) = 2^{n-\frac{7}{2}}(\eta-\xi)(2-\eta-\xi)^{\frac{1}{2}-n}\left[(1-\eta)(1-\xi)\right]^{n-\frac{3}{2}} \in C^{n-2}(\bar{D}_\varepsilon^{(1)}). \tag{6.43}$$

From (6.10) and (6.43) it follows that $c(\xi,\eta) \le 0$, $g(\xi,\eta) \ge 0$ in $\bar{D}_\varepsilon^{(1)}$ for $\varepsilon \in (0,1)$. Hence Theorem 3.1 and Lemma 3.1 imply

Proposition 6.2. *There exists a classical solution* $U(\xi,\eta) \in C^{n-1}(\bar{D}_0 \setminus (1,1))$ *for the problem* (6.41), (6.42) *for which*

$$U(\xi,\eta) \geq 0, \quad U_\eta(\xi,\eta) \geq 0, \quad U_\xi(\xi,\eta) \geq 0 \quad in \ \bar{D}^{(1)}_\varepsilon.$$

A elementary calculation shows that $g(\xi,\xi) = 0$,

$$g_\eta(\xi,\xi) = -g_\xi(\xi,\xi) = \frac{1}{8}(1-\xi)^{n-\frac{5}{2}} \geq 0 \tag{6.44}$$

$$g_{\xi\eta}(\xi,\eta) + c(\xi,\eta)g(\xi,\eta) = 0. \tag{6.45}$$

Since

$$g_\eta(\xi,\eta) = g(\xi,\eta)\left[\frac{1}{\eta-\xi} + \frac{n-\frac{1}{2}}{2-\eta-\xi} - \frac{n-\frac{3}{2}}{1-\eta}\right]$$

and

$$g_\eta(1-\varepsilon,\eta) = \frac{\varepsilon g(1-\varepsilon,\eta)}{(1-\eta)(\varepsilon^2-(1-\eta)^2)}\left[\frac{1}{2} + n - \eta(\frac{1}{2}+n) + \varepsilon(\frac{3}{2}-n)\right],$$

for

$$\eta_\varepsilon = 1 - \varepsilon\frac{2n-3}{2n+1}$$

we have

$$\begin{aligned} g_\eta(1-\varepsilon,\eta) &> 0 \text{ for } 1-\varepsilon < \eta < \eta_\varepsilon, && (6.46)\\ g_\eta(1-\varepsilon,\eta) &< 0 \text{ for } \eta_\varepsilon < \eta < 1. && (6.47)\end{aligned}$$

To show (6.38), let

$$K_2 = \int_{\bar{D}^{(1)}_{\frac{1}{2}}} g^2(\xi,\eta)\,d\xi\,d\eta > 0.$$

Then

$$0 < K_2 \leq \int_{\bar{D}^{(1)}_\varepsilon} g^2(\xi,\eta)\,d\xi\,d\eta, \qquad 0 < \varepsilon < \frac{1}{2}.$$

Using arguments similar to that of Theorem 6.1, we arrive to (6.18). By (6.45), we get

$$\begin{aligned} 0 < K_2 \leq \int_{D^{(1)}_\varepsilon} g^2(\xi,\eta)\,d\xi\,d\eta &= -\int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\,d\eta. \\ &- \int_0^{1-\varepsilon} [U_\xi(\xi,\xi)g(\xi,\xi) + U(\xi,\xi)g_\eta(\xi,\xi)]\,d\xi \end{aligned}$$

Since $g(\xi,\xi) = 0$, the above inequality becomes

$$\begin{aligned} 0 < K_2 \leq -\int_0^{1-\varepsilon} U(\xi,\xi)g_\eta(\xi,\xi)\,d\xi &- \int_{1-\varepsilon}^{\eta_\varepsilon} U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\,d\eta \\ &- \int_{\eta_\varepsilon}^{1} U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)\,d\eta. \end{aligned}$$

Following the steps of the proof of Theorem 6.1 and using the Proposition 6.2, we find

$$\begin{aligned}0 < K_2 &\leq \int_{\eta_\varepsilon}^{1} U(1-\varepsilon,\eta)|g_\eta(1-\varepsilon,\eta)|\,d\eta - \int_{1-\varepsilon}^{\eta_\varepsilon} U(1-\varepsilon,\eta)|g_\eta(1-\varepsilon,\eta)|\,d\eta \\ &\leq \int_{\eta_\varepsilon}^{1} U(1-\varepsilon,1)|g_\eta(1-\varepsilon,\eta)|\,d\eta - \int_{1-\varepsilon}^{\eta_\varepsilon} U(1-\varepsilon,1-\varepsilon)|g_\eta(1-\varepsilon,\eta)|\,d\eta \\ &= [U(1-\varepsilon,1) - U(1-\varepsilon,1-\varepsilon)]\,g(1-\varepsilon,\eta_\varepsilon).\end{aligned}$$

By (6.43), it follows that

$$g(1-\varepsilon,\eta_\varepsilon) \leq \varepsilon^{n-\frac{3}{2}}$$

and so

$$0 < K_2 \leq [U(1-\varepsilon,1) - U(1-\varepsilon,1-\varepsilon)]\,\varepsilon^{n-\frac{3}{2}}.$$

Finally, using (6.9), it follows that

$$0 < K_2 \leq \left[u_n^{(2)}\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right) - u_n^{(2)}(\varepsilon,0)\right]\varepsilon^{n-\frac{3}{2}},$$

i.e.

$$u_n^{(1)}(\varrho,\varrho) \geq u_n^{(1)}(2\varrho,0) + K_2\varrho^{1-n} \geq K_3\varrho^{1-n}, K_3 = 2^{1-n}K_2,$$

and so the estimate (6.38) holds. The proof of Theorem 6.2 is complete.■

REMARK 6.1. *In* [**2**], *Theorem 2, Aldashev considers the following type problems:*

Find a solution of the homogeneous wave equation $\Box u = 0$ in Ω_0, satisfying *the nonhomogeneous boundary conditions:*

$P1'$: $u|_{\Sigma_0} = \tau_0(x)$, $u|_{\Sigma_1} = \sigma_1(x)$ or

$P2'$: $u_t|_{\Sigma_0} = \nu_0(x)$, $u|_{\Sigma_1} = \sigma_1(x)$.

Under certain conditions, imposed on the functions τ_0, σ_1, ν_0, he asserts that both Problems $P1'$ and $P2'$ are solvable in the class $C(\bar{\Omega}_0) \cap C^2(\Omega_0)$.

Comparing these conclusion with Theorems 6.1, 6.2 and the results presented in [**21**], it is not difficult to see the appearing contradiction. Indeed, applying the Duhamel's formula to the nonhomogeous wave equation (6.6) in Ω_0 with homogeneous Cauchy initial dates on Σ_0, we find the solution of this problem in $C^{n-1}(\bar{\Omega}_0)$, expressed by explicit formulas (see, [**23**], pp. 226-234). Therefore, the problem (6.6), (6.7) transforms to the problem $P2'$ with $v(x) \equiv 0$ and $\sigma_1 \in C^{n-1}(\bar{\Sigma}_0)$. But the last problem cannot be solved in $C(\bar{\Omega}_0)$, because, by Theorem 6.1, for $\alpha \equiv 0$ the unique generalized solution of Problem P_α has a power-type singularity of the form ρ^{-n} (see, (6.3)) at the point $(0,0,0)$.

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OSCILLATION OF SECOND ORDER LINEAR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In the present paper the problem of oscillation of all solutions of the second order linear delay equation

$$u''(t) + p(t)u(\tau(t)) = 0$$

is investigated, where p is a nonnegative locally summable function. For this equation a general oscillation criterion is obtained showing the joint contribution of the following two factors: the presence of the delay and the second order nature of the equation. Using this criterion, effective sufficient oscillation conditions are derived. Some of them concern delay equations only, and others involve ordinary differential equations as well . A number of known results, in particular a generalization of well-known Hille's criteria to delay equations, are improved. Several examples illustrate that some of the results obtained are best possible in a sense.

1. INTRODUCTION

Consider the linear second order delay equation

$$u''(t) + p(t)u(\tau(t)) = 0, \tag{1.1}$$

where $p : R_+ \to R_+$ is locally integrable, $\tau : R_+ \to R$ is continuous, $\tau(t) \le t$ for $t \ge 0$, $\tau(t) \to +\infty$ as $t \to +\infty$ and

$$\text{mes}\,\{s \ge t : p(s) > 0\} > 0 \qquad \text{for } t \ge 0, \tag{1.2}$$

where mes denotes the Lebesgue measure on the real line. These assumptions will be supposed to hold throughout the paper.

Let $T_0 = \min\{\tau(t) : t \ge 0\}$ and

$$\tau_{(-1)}(t) = \sup\{s \ge 0 : \tau(s) \le t\} \quad \text{for} \quad t \ge T_0.$$

Clearly $\tau_{(-1)}(t) \ge t$ for $t \ge T_0$, $\tau_{(-1)}$ is nondecreasing and coincides with the inverse of τ when the latter exists. Besides, put $\tau_{(-2)} = \tau_{(-1)} \circ \tau_{(-1)}$.

A continuous function $u : [t_0, +\infty[\to R$ is said to be a *solution* of (1.1) if it is locally absolutely continuous on $[\tau_{(-1)}(t_0), +\infty[$ along with its derivative and almost everywhere on $[\tau_{(-1)}(t_0), +\infty[$ satisfies (1.1). A solution of (1.1) is said to be *proper* if it is not identically zero in any neighbourhood of $+\infty$. A proper solution is called

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oscillatory (or it is said to oscillate) if it has a sequence of zeros tending to $+\infty$. Otherwise it is called *nonoscillatory.*

We say that the equation (1.1) is *oscillatory* if each one of its proper solutions oscillates. Otherwise we call (1.1) *nonoscillatory.*

The present paper is devoted to the problem of oscillation of (1.1). For the case of ordinary differential equations, i.e. when $\tau(t) \equiv t$, the history of the problem began as early as in 1836 by the work of Sturm [16] and was continued in 1893 by A. Kneser [11]. Essential contribution to the subject was made by E. Hille, A. Wintner, Ph. Hartman, W. Leighton, Z. Nehari, and others (see the monograph by C. Swanson [17] and the references cited therein). In particular, in 1948 E. Hille [6] obtained the following well-known oscillation criteria.

Let

$$\limsup_{t\to+\infty} t\int_t^{+\infty} p(s)ds > 1 \tag{1.3}$$

or

$$\liminf_{t\to+\infty} t\int_t^{+\infty} p(s)ds > \frac{1}{4}, \tag{1.4}$$

the conditions being assumed to be satisfied if the integral diverges. Then (1.1) with $\tau(t) \equiv t$ *is oscillatory.*

For the delay differential equation (1.1) earlier oscillation results can be found in the monographs by A. Myshkis [14] and S. Norkin [15]. In 1968 P. Waltman [19] and in 1970 J. Bradley [1] proved that (1.1) is oscillatory if $\int^{+\infty} p(t)dt = +\infty$. Proceeding in the direction of generalization of Hille's criteria, in 1971 J.Wong [21] showed that if $\tau(t) \geq \alpha t$ for $t \geq 0$ with $0 < \alpha \leq 1$, then the condition

$$\liminf_{t\to+\infty} t\int_t^{+\infty} p(s)ds > \frac{1}{4\alpha} \tag{1.5}$$

is sufficient for the oscillation of (1.1). In 1973 L. Erbe [2] generalized this condition to

$$\liminf_{t\to+\infty} t\int_t^{+\infty} \frac{\tau(s)}{s}p(s)ds > \frac{1}{4} \tag{1.6}$$

without any additional restriction on τ. In 1987 J. Yan [18] obtained some general criteria improving the previous ones.

An oscillation criterion of different type is given in 1986 by R. Koplatadze [7] and in 1988 by J. Wei [20], where it is proved that (1.1) is oscillatory if

$$\limsup_{t\to+\infty} \int_{\tau(t)}^t \tau(s)p(s)ds > 1 \tag{1.7}$$

or

$$\liminf_{t\to+\infty} \int_{\tau(t)}^t \tau(s)p(s)ds > \frac{1}{e}. \tag{1.8}$$

The conditions (1.7) and (1.8) are analogous to the oscillation conditions due to Ladas, Lakshmikantham and Papadakis [13], and Koplatadze and Chanturia [9], respectively,

$$L := \limsup_{t\to+\infty} \int_{\tau(t)}^{t} p(s)ds > 1, \tag{1.9}$$

$$l := \liminf_{t\to+\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e} \tag{1.10}$$

for the first order delay equation

$$u'(t) + p(t)u(\tau(t)) = 0. \tag{1.11}$$

The essential difference between (1.5)–(1.6) and (1.7)–(1.8) is that the first two can guarantee oscillation for ordinary differential equations as well, while the last two work only for delay equations. Unlike first order differential equations, where the oscillatory character is due to the delay only, the equation (1.1) can be oscillatory without any delay at all, i.e., in the case $\tau(t) \equiv t$. Figuratively speaking, two factors contribute to the oscillatory character of (1.1): the presence of the delay and the second order nature of the equation. The conditions (1.5)–(1.6) and (1.7)–(1.8) illustrate the role of these factors taken separately.

In the present paper, developing the ideas of [7], we obtain integral oscillation criteria for (1.1) where the joint contribution of the above mentioned factors is presented. These criteria are formulated in terms of solutions of certain integral inequalities and enable us to obtain new effective sufficient conditions for the oscillation of (1.1) generalizing (1.5)–(1.8) not only in the case of delay equations, but for ordinary differential equations as well. Several examples illustrate their worth.

In Section 2 a number of lemmas is given showing consecutive steps of our reasoning. Section 3 is dedicated to oscillation criteria caused by the presence of the delay. We show that these criteria have essentially first order character by reducing the problem of oscillation of (1.1) to that of a first order delay differential equation. In Section 4 we formulate a general oscillation theorem and some of its corollaries more convenient for obtaining effective sufficient conditions. In section 5 we obtain Hille type effective oscillation conditions for (1.1) which are due to its second order nature.

In what follows it will be assumed that the condition

$$\int^{+\infty} \tau(s)p(s)ds = +\infty \tag{1.12}$$

is fulfilled. As it follows from Lemma 4.1 in [8], this condition is necessary for (1.1) to be oscillatory. The paper being devoted to the problem of oscillation of (1.1), the condition (1.12) does not affect the generality.

2. Preliminary lemmas

Lemma 2.1. *Let* (1.12) *be fulfilled,* $u : [t_0, +\infty[\to]0, +\infty[$ *be a positive solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$. *Then*
(i) $u'(t) > 0, \ \ u(t) \geq t\,u'(t)$ *for* $t \geq T$;
(ii) u *is nondecreasing on* $[T, +\infty[$, *while the function* $t \mapsto u(t)/t$ *is nonincreasing*

on $[T, +\infty[$*;*
(iii) *for any function* $\nu : R_+ \to R$ *satisfying*

$$\nu(t) \le t \quad for \quad t \in R_+, \qquad \nu(t) \to +\infty \quad as \quad t \to +\infty, \tag{2.1}$$

we have

$$u(\tau(t)) \ge \tau_{/\nu}(t)\, u(\nu(t)) \quad for \quad t \ge \max\{T, \nu_{(-1)}(t_0)\}, \tag{2.2}$$

where

$$\tau_{/\nu}(t) = \begin{cases} 1 & if \ \tau(t) \ge \nu(t), \\ \frac{\tau(t)}{\nu(t)} & if \ \tau(t) \le \nu(t). \end{cases} \tag{2.3}$$

Proof. In view of (1.2) it is obvious that $u'(t) > 0$ for $t \ge 0$. Let $\rho(t) \equiv u(t) - t\,u'(t)$. Since $\rho'(t) = -t\,u''(t) \ge 0$ for $t \ge T$, we have either $u(t) - t\,u'(t) \ge 0$ for $t \ge T$ or $u(t) - t\,u'(t) < 0$ for $t \ge t_1$ with some $t_1 \ge T$. To prove (i), it suffices to show that the latter is impossible. Indeed, otherwise

$$\left(\frac{u(t)}{t}\right)' = \frac{t\,u'(t) - u(t)}{t^2} > 0 \quad \text{for} \quad t \ge t_1,$$

whence $u(\tau(t)) \ge c\,\tau(t)$ for $t \ge t_2 = \tau_{(-1)}(t_1)$ with some $c > 0$. The equation (1.1) then yields

$$u'(t_2) \ge \int_{t_2}^{+\infty} p(s)u(\tau(s))ds \ge c \int_{t_2}^{+\infty} p(s)\tau(s)ds$$

which contradicts (1.12). Thus (i) is proved. (ii) is an immediate consequence of (i), and (iii) follows from (ii). The proof is complete.

Remark 2.1. Without the condition (1.12) the following weaker versions of (i) and (iii) are valid (see [20], Lemma 1 and [2], Lemma 2.1, respectively): for each $0 < \gamma < 1$ there is $T_\gamma \ge T$ such that $u(t) \ge \gamma t\,u'(t)$ and $u(\tau(t)) \ge \gamma\tau_{/\nu}(t)\,u(\nu(t))$ for $t \ge T_\gamma$. It should be noted that in the applications below these versions would be sufficient.

Lemma 2.1 (i) implies

$$u(\tau(t)) \ge \tau(t)\,u'(\tau(t)) \quad \text{for} \quad t \ge T. \tag{2.4}$$

This inequality, however, can be improved.

Lemma 2.2. *Let* (1.12) *be fulfilled,* $u : [t_0, +\infty[\to]0, +\infty[$ *be a positive solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$*. Then*

$$u(\tau(t)) \ge \tau_T(t)u'(\tau(t)) \quad for \quad t \ge \tau_{(-1)}(T), \tag{2.5}$$

where

$$\tau_T(t) = \tau(t) + \int_T^{\tau(t)} \xi\tau(\xi)p(\xi)d\xi \quad for \quad t \ge \tau_{(-1)}(T). \tag{2.6}$$

Proof. Integrate the identity $(u(t) - t\,u'(t))' = t\,p(t)\,u(\tau(t))$ from T to $\tau(t) \geq T$ and use (2.5) to get

$$u(\tau(t)) \geq \tau(t)u'(\tau(t)) + \int_T^{\tau(t)} \xi p(\xi)u(\tau(\xi))d\xi \quad \text{for} \quad t \geq \tau_{(-1)}(T).$$

To estimate the last integral, use Lemma 2.1(iii) with $\nu(t) \equiv t$, Lemma 2.1 (i) and the nondecreasing character of u'. We get

$$\int_T^{\tau(t)} \xi p(\xi)u(\tau(\xi))d\xi \geq \int_T^{\tau(t)} \tau(\xi)p(\xi)u(\xi)d\xi \geq \int_T^{\tau(t)} \xi\tau(\xi)p(\xi)u'(\xi)d\xi \geq$$
$$\geq \left(\int_T^{\tau(t)} \xi\tau(\xi)p(\xi)d\xi\right) u'(\tau(t)) \quad \text{for} \quad t \in \tau_{(-1)}(T).$$

The last two inequalities imply (2.5). The proof is complete.

Lemma 2.2 immediately implies

Lemma 2.3. *Let* (1.12) *be fulfilled,* $u : [t_0, +\infty[\to]0, +\infty[$ *be a positive solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$. *Then the function* $x : [T, +\infty[\to]0, +\infty[$ *defined by* $x(t) = u'(t)$ *is a positive solution of the differential inequality*

$$x'(t) + \tau_T(t)x(\tau(t)) \leq 0, \tag{2.7}$$

where τ_T *is defined by* (2.6).

The estimate (2.5) is essential for the results of Section 3. Being more exact than (2.4), via Lemma 2.3 it will enable us to improve the criteria (1.7) and (1.8).

The following four lemmas are crucial in proving the general oscillation theorem in Section 4, especially Lemmas 2.5 and 2.7 giving important estimates. Note beforehand that a continuous function $v : [T, +\infty[\to]0, +\infty[$ ($w : [T, +\infty[\to]0, +\infty[$) is a solution of the integral inequality (2.8) (integral inequality (2.11)) if it satisfies (2.8) ((2.11)) for $t \geq \tau_{(-2)}(T)$ ($t \geq \nu_{(-1)}(T)$). The same is true for the integral equations (2.17) and (2.18). Note also that solutions of these integral inequalities and equations are necessarily positive.

Lemma 2.4. *Let* (1.12) *be fulfilled,* $u : [t_0, +\infty[\to]0, +\infty[$ *be a positive solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$. *Then the function* $v : [T, +\infty[\to]0, +\infty[$ *defined by* $v(t) = \frac{u'(\tau(t))}{u'(t)}$ *is a solution of the integral inequality*

$$v(t) \geq \exp\left\{\int_{\tau(t)}^{t} \tau_T(\xi)\,p(\xi)v(\xi)\,d\xi\right\}, \quad t \geq \tau_{(-2)}(T). \tag{2.8}$$

Proof. We have $v(t) = \frac{x(\tau(t))}{x(t)}$ for $t \geq T$, where, according to Lemma 2.3, x is a positive solution of (2.7). If we rewrite (2.7) as

$$\frac{x'(t)}{x(t)} \leq -\tau_T(t)\,p(t)\,v(t) \quad \text{for} \quad t \geq \tau_{(-1)}(T) \tag{2.9}$$

and integrate from t to $\tau(t)$, then we get (2.8) thus completing the proof.

Lemma 2.5. *Let* (1.12) *be fulfilled,* $u : [t_0, +\infty[\to]0, +\infty[$ *be a positive solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$. *Then there exists a solution* $v : [T, +\infty[\to]0, +\infty[$ *of* (2.8) *such that*

$$u'(s) \geq \exp\left\{\int_s^t \tau_T(\xi)p(\xi)\,v(\xi)\,d\xi\right\} u'(t) \quad for \quad t \geq s \geq \tau_{(-1)}(T). \tag{2.10}$$

Proof. By (2.9)

$$\frac{u''(t)}{u'(t)} \leq -\tau_T(t)p(t)v(t) \quad \text{for} \quad t \geq \tau_{(-1)}(T)$$

where v is a solution of the (2.8). Integrating this inequality from t to s, we get (2.10) thus completing the proof.

Lemma 2.6. *Let a continuous function* $\nu : R_+ \to R$ *satisfy* (2.1), $u : [t_0, +\infty[\to]0, +\infty[$ *be a positive solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$. *Then the function* $w : [T, +\infty[\to]0, +\infty[$ *defined by* $w(t) = \frac{u(\nu(t))}{u'(t)}$ *is a solution of the integral inequality*

$$w(t) \geq \int_T^{\nu(t)} \left\{\exp \int_s^t \tau_{/\nu}(\xi)p(\xi)w(\xi)\,d\xi\right\} ds, \quad t \geq \nu_{(-1)}(T), \tag{2.11}$$

where $\tau_{/\nu}$ *is defined by* (2.3).

Proof. If we write (1.1) as

$$(u'(t))' = -p(t)\frac{u(\tau(t))}{u'(t)}\,u'(t) \quad \text{for} \quad t \geq T, \tag{2.12}$$

then we have

$$u'(t) = u'(T)\exp\left\{-\int_T^t p(\xi)\,\frac{u(\tau(\xi))}{u'(\xi)}\,d\xi\right\} ds \quad \text{for} \quad t \geq T, \tag{2.13}$$

$$u(\nu(t)) \geq u'(T)\int_T^{\nu(t)} \exp\left\{-\int_T^s p(\xi)\,\frac{u(\tau(\xi))}{u'(\xi)}d\xi\right\} ds \quad \text{for } t \geq \nu_{-1}(T). \tag{2.14}$$

Dividing (2.14) by (2.13) and using (2.2), we get (2.11). The proof is complete.

Lemma 2.7. *Let a continuous function* $\nu : R_+ \to R$ *satisfy* (2.1), $u : [t_0, +\infty[\to]0, +\infty[$ *be a solution of* (1.1) *and* $T = \tau_{(-1)}(t_0)$. *Then there exists a solution* $w : [T, +\infty[\to]0, +\infty[$ *of* (2.11) *such that*

$$u(t) \geq \left(t + \int_T^t s\,\tau_{/\nu}(s)p(s)\,w(s)\,ds\right) u'(t) \quad for \quad t \geq T. \tag{2.15}$$

Proof. Integrate the identity $(u(t) - t\,u'(t))' = t\,p(t)\,u(\tau(t))$ from T to $t \geq T$ and use (2.2) to get

$$\begin{aligned} u(t) \;&\geq\; t\,u'(t) + \int_T^t s\,p(s)\frac{u(\tau(s))}{u'(s)}u'(s)\,ds \geq \\ &\geq\; \left(t + \int_T^t s\,\tau_{/\nu}(s)p(s)\,w(s)\,ds\right) u'(t) \;\text{ for }\; t \geq T, \end{aligned} \tag{2.16}$$

where, according to Lemma 2.6, w is a solution of (2.11). Thus (2.15) holds and the proof is complete.

Since (1.12) is necessary for the oscillation of (1.1), its violation via Lemmas 2.4 and 2.6 imply the existence of solutions of (2.8) and (2.11). The following two lemmas give more exact results which will permit us to do without the condition (1.12) in Section 4.

Lemma 2.8. *Let* (1.12) *be violated. Then the integral equation corresponding to* (2.8)

$$v(t) = \exp\left\{\int_{\tau(t)}^{t} \tau_T(s)\, p(s) v(s)\, ds\right\} \tag{2.17}$$

has a bounded solution.

Proof. Let $M > 1$ be an arbitrary number. There exists $\delta > 0$ such that $\exp(\delta M) \le M$. Since (1.12) is violated, there exists $T_0 \ge 0$ such that $\int_{T_0}^{+\infty}(L+1)\tau(s)p(s)ds \le \delta$, where $L = \int_0^{+\infty} \tau(s)p(s)ds$. We claim that for any $T \ge T_0$ (2.17) has a solution v satisfying $1 \le v(t) \le M$ for $t \ge T$. To show this, consider the bounded convex closed set $V = \{v \in C([T, +\infty[) : 1 \le v(t) \le M\}$ in the space $C([T, +\infty[)$ of all continuous on $[T, +\infty[$ functions with the topology of uniform convergence on every finite interval, and consider the operator Q on V defined by

$$Q(v)(t) = \begin{cases} \exp\left\{\int_{\tau(t)}^{t} \tau_T(s)\, p(s)\, v(s)\, ds\right\} & \text{for } t \ge \tau_{(-2)}(T), \\ Q(v)(\tau_{(-2)}(T)) & \text{for } T_0 \le t \le \tau_{(-2)}(T). \end{cases}$$

Since

$$\int_T^{\tau(s)} \xi\tau(\xi)p(\xi)d\xi \le \tau(s)\int_0^{\tau(s)} \tau(\xi)p(\xi)d\xi \le L\tau(s),$$

it can be easily checked that Q maps V into itself and satisfies all the conditions of the Schauder-Tychonoff fixed point theorem (see, e.g., [3], pp.161–163). The fixed point of Q obviously is a solution of (2.17). The proof is complete.

Lemma 2.9. *Let* (1.12) *be violated and a continuous function* $\nu : R_+ \longrightarrow R$ *satisfy* (2.1). *Then for all sufficiently large* T *the integral equation corresponding to* (2.11)

$$w(t) = \int_T^{\nu(t)} \exp\left\{\int_s^t \tau_{/\nu}(\xi)\, p(\xi)\, w(\xi)\, d\xi\right\} ds \tag{2.18}$$

has a solution w *such that* w/ν *is bounded.*

Proof. Let $M > 1$, $\delta > 0$ and $T_0 \ge 0$ be as in the proof of Lemma 2.8. Then for any $T \ge T_0$, (2.18) has a solution w satisfying $\nu(t) \le w(t) \le M\nu(t)$ for $t \ge T_0$. Indeed, using the inequality $\tau_{/\nu}(t)\,\nu(t) \le \tau(t)$, we get convinced that the set $V = \{w \in C([T, +\infty[) : \nu(t) \le w(t) \le M\,\nu(t)\}$ and the operator Q defined on V by

$$Q(w)(t) = \begin{cases} \int_T^{\nu(t)} \exp\left\{\int_s^t \tau_{/\nu}(\xi)\, p(\xi)\, w(\xi)\, d\xi\right\} ds & \text{for } t \ge v_{(-1)}(T), \\ Q(w)(\tau_{(-1)}(T)) & \text{for } T \le t \le \nu_{(-1)}(T) \end{cases}$$

satisfy all the conditions of the Schauder-Tychonoff fixed point theorem. As above, the fixed point of Q is a solution of (2.18). The proof is complete.

3. Oscillations Caused by the Delay

In this section oscillation results are obtained for (1.1) by reducing it to a first order equation. Since for the latter the oscillation is due solely to the delay, the criteria hold for delay equations only and do not work in the ordinary case. The section is independent of the general oscillation Theorem 4.1 and is based only on Lemma 2.3. It should be observed, however, that by means of lower a priori asymptotic estimates for v (as in Section 5 for w) Theorems 3.3 and 3.5 (unlike Theorem 3.4) could be deduced from Corollary 4.2 below.

Lemma 2.3 immediately implies

Theorem 3.1. *Let* (1.12) *be fulfilled and the differential inequality* (2.7) *have no eventually positive solution. Then the equation* (1.1) *is oscillatory.*

Theorem 3.1 reduces the question of oscillation of (1.1) to that of the absence of eventually positive solutions of the differential inequality

$$x'(t) + \left(\tau(t) + \int_T^{\tau(t)} \xi\tau(\xi)p(\xi)d\xi\right) p(t)x(\tau(t)) \le 0. \tag{3.1}$$

So oscillation results for first order delay differential equations can be applied since the oscillation of the equation

$$u'(t) + g(t)u(\delta(t)) = 0 \tag{3.2}$$

is equivalent to the absence of eventually positive solutions of the inequality

$$u'(t) + g(t)u(\delta(t)) \le 0. \tag{3.3}$$

This fact is a simple consequence of the following comparison theorem deriving the oscillation of (3.2) from the oscillation of the equation

$$v'(t) + h(t)v(\sigma(t)) = 0. \tag{3.4}$$

We assume that $g, h : R_+ \to R_+$ are locally integrable, $\delta, \sigma : R_+ \to R$ are continuous, $\delta(t) \le t$, $\sigma(t) \le t$ for $t \in R_+$, and $\delta(t) \to +\infty$, $\sigma(t) \to +\infty$ as $t \to +\infty$.

Theorem 3.2. *Let*

$$g(t) \ge h(t) \ \ \textit{and} \ \ \delta(t) \le \sigma(t) \ \ \textit{for} \ t \in R_+, \tag{3.5}$$

and let the equation (3.4) *be oscillatory. Then* (3.2) *is also oscillatory.*

Corollary 3.1. *Let the equation* (3.2) *be oscillatory. Then the inequality* (3.3) *has no eventually positive solution.*

Proof. Suppose, to the contrary, that there exists a positive solution $u: [t_0, +\infty[\to R$ of (3.3). Then u is a solution of the equation $v'(t) + h(t)v(\delta(t)) = 0$, where $h(t) \equiv -\frac{u'(t)}{u(\delta(t))} \geq g(t)$. According to Theorem 3.2, the equation (3.2) must have a nonoscillatory solution which contradicts to the hypothesis of the corollary.

In the case $\delta(t) \equiv \sigma(t)$ Theorem 3.2 can be found in [5] (Theorem 3.1), and in the general case but under the additional restriction $\sigma(t) < t$ in [12] (Theorem 2.8). Since these restrictions are not imposed here, we present the proof, which, in our opinion, is interesting by itself.

Proof of Theorem 3.2. Let, to the contrary of the assertion of the theorem, (3.2) have a nonoscillatory solution $u : [t_0, +\infty[\to R$ which is supposed to be positive. In the space of all continuous on $[t_0, +\infty[$ functions with the topology of locally uniform convergence consider the set V consisting of all continuous $v : [t_0, +\infty[\to R$ satisfying

$$v(t) = u(t_0) \quad \text{for} \quad t_0 \leq t \leq T,$$

$$u(t) \leq v(t) \leq u(t_0) \quad \text{for} \quad t \geq T, \tag{3.6}$$

$$1 \leq \frac{v(\sigma(t))}{v(t)} \leq \frac{u(\delta(t))}{u(t)} \quad \text{for} \quad t \geq T, \tag{3.7}$$

where $T = \delta_{(-1)}(t_0)$. V is nonempty ($u \in V$) and bounded. Moreover, it is convex since

$$\frac{\lambda v_1(\sigma(t)) + (1-\lambda) v_2(\sigma(t))}{\lambda v_1(t) + (1-\lambda) v_2(t)} = \frac{\lambda}{v_2(t)} \left[\frac{\lambda}{v_2(t)} + \frac{1-\lambda}{v_1(t)}\right]^{-1} \frac{v_1(\sigma(t))}{v_1(t)} +$$

$$+ \frac{1-\lambda}{v_1(t)} \left[\frac{\lambda}{v_2(t)} + \frac{1-\lambda}{v_1(t)}\right]^{-1} \frac{v_2(\sigma(t))}{v_2(t)} \quad \text{for} \quad v_1,\, v_2 \in V, \ t \geq T.$$

Define the operator Q on V by

$$Q(v)(t) = \begin{cases} u(t_0) \exp\left\{-\int_{t_0}^t h(s) \frac{v(\sigma(s))}{v(s)} ds\right\} & \text{for} \quad t \geq T, \\ u(t_0) & \text{for} \quad t_0 \leq t \leq T. \end{cases}$$

Clearly $Q(v)(t) \leq u(t_0)$ for $t \geq t_0$. On the other hand, by (3.5) and (3.7) we get

$$Q(v)(t) \geq u(t_0) \exp\left\{-\int_{t_0}^t g(s) \frac{u(\delta(s))}{u(s)} ds\right\} = u(t) \quad \text{for} \quad t \geq T,$$

so (3.6) is fulfilled with Qv instead of v. The same is true for (3.7) since, by (3.5) and (3.7), we have

$$\begin{aligned} 1 \ &\leq \ \frac{Q(v)(\sigma(t))}{Q(v)(t)} = \exp\left\{\int_{\sigma(t)}^t h(s) \frac{v(\sigma(s))}{v(s)} ds\right\} \leq \\ &\leq \ \exp\left\{\int_{\delta(t)}^t g(s) \frac{u(\delta(s))}{u(s)} ds\right\} = \frac{u(\delta(t))}{u(t)} \quad \text{for} \quad t \geq T. \end{aligned}$$

Thus $QV \subset V$. Besides, standard arguments show that T is completely continuous in the topology of uniform convergence on every finite segment. Hence the Schauder-Tychonoff fixed point theorem implies the existence of v_0 such that $Qv_0 = v_0$ which obviously is a nonoscilatory solution of (3.4). The obtained contradiction proves the theorem.

Turning to applications of Theorem 3.1, we will use it together with the criteria (1.9) and (1.10) to get

Theorem 3.3. *Let*

$$K := \limsup_{t\to+\infty} \int_{\tau(t)}^{t} \left(\tau(s) + \int_0^{\tau(s)} \xi\tau(\xi)p(\xi)d\xi\right) p(s)ds > 1 \tag{3.8}$$

or

$$k := \liminf_{t\to+\infty} \int_{\tau(t)}^{t} \left(\tau(s) + \int_0^{\tau(s)} \xi\tau(\xi)p(\xi)d\xi\right) p(s)ds > \frac{1}{e}. \tag{3.9}$$

Then the equation (1.1) *is oscillatory.*

To apply Theorem 3.1, it suffices to note that: (i) (1.12) is fulfilled since otherwise $k = K = 0$; (ii) since $\tau(t) \to +\infty$ as $t \to +\infty$, the relations (3.8)-(3.9) imply the same relations with 0 changed by any $T \geq 0$.

Remark 3.1. Theorem 3.3 improves the criteria (1.7)–(1.8) of R. Koplatadze [7] and J. Wei [20] mentioned in the introduction. This is directly seen from (3.8)–(3.9) and can be easily checked if we take $\tau(t) \equiv t - \tau_0$ and $p(t) \equiv p_0/(t-\tau_0)$ for $t \geq 2\tau_0$, where the constants $\tau_0 > 0$ and $p_0 > 0$ satisfy $\tau_0 p_0 < 1/e$. In this case neither of (1.7)–(1.8) is applicable for (1.1) while both (3.8)–(3.9) give the positive conclusion about its oscillation. Note also that this is exactly the case where the oscillation is due to the delay since the corresponding equation without delay is nonoscillatory.

Remark 3.2. The criteria (3.8)–(3.9) look like (1.9)–(1.10), but there is an essential difference between them pointed out in the introduction. The condition (1.10) is close to the necessary one since according to [9] if $L \leq 1/e$, then (3.2) is nonoscillatory. On the other hand, for an oscillatory (1.1) without delay we have $k = K = 0$. Nevertheless, the constant $1/e$ in Theorem 3.3 is also best possible in the sense that for any $\varepsilon \in]0, 1/e]$ it can not be replaced by $1/e - \varepsilon$ without affecting the validity of the theorem. This is illustrated by the following

Example 3.1. Let $\varepsilon \in]0, 1/e]$, $1 - e\varepsilon < \beta < 1$, $\tau(t) \equiv \alpha t$ and $p(t) \equiv \beta(1-\beta)\alpha^{-\beta}t^{-2}$, where $\alpha = e^{\frac{1}{\beta-1}}$. Then (3.9) is fulfilled with $1/e$ replaced by $1/e - \varepsilon$. Nevertheless (1.1) has a nonoscillatory solution, namely $u(t) \equiv t^\beta$. Indeed, denoting $c = \beta(1-\beta)\alpha^{-\beta}$, we see that the expression under the limit sign in (3.9) is constant and equals $\alpha c |\ln \alpha| (1 + \alpha c) = (\beta/e)(1 + (\beta(1-\beta))/e) > \beta/e > 1/e - \varepsilon$.

There is a gap between the conditions (1.9)–(1.10) and (3.8)–(3.9) when $0 \leq l \leq 1/e$, $l < L$, and $0 \leq k \leq 1/e$, $k < K$, respectively. In the case of first order equations there arises an interesting problem of filling this gap, i.e. of finding of a function $f : [0, 1/e] \longrightarrow [1/e, 1]$ such that the condition $L > f(l)$ would guarantee the oscillation of (3.2). Moreover, it makes sense to seek for an optimal function in the sense that $L < f(l)$ would imply nonoscillation. A number of papers are devoted to this problem (see, for example, [4] and the references therein). Using results in this direction, one can derive various sufficient conditions for the oscillation of (1.1).

According to Remark 3.1, neither of them can be optimal in the above sense but nevertheless they are of interest since they cannot be derived from Corollary 4.2 of the general oscillation theorem. We combine Theorem 3.1 with the best to our knowledge result in this direction ([4], Corollary 1) to obtain

Theorem 3.4. *Let K and k be defined by* (3.8)–(3.9), $0 \le k \le 1/e$ *and*

$$K > k + \frac{1}{\lambda(k)} - \frac{1 - k - \sqrt{1 - 2k - k^2}}{2},$$

where $\lambda(k)$ is the smaller root of the equation

$$\lambda = \exp(k\lambda). \tag{3.10}$$

Then (1.1) *is oscillatory.*

Finally we give a criterion which follows from Theorem 3.1 and a simplified version of Theorem 3 in [10]. For the sake of simplicity we will formulate the theorem in terms of τ_0 (see (2.6)).

Theorem 3.5. *Let k be defined by* (3.9), $0 \le k \le 1/e$ *and*

$$\limsup_{t \to +\infty} \int_{\delta(t)}^{t} p(s)\tau_0(s) \exp\left(\lambda(k) \int_{\delta(s)}^{\delta(t)} p(\xi)\tau_0(\xi) d\xi \right) ds > 1,$$

where $\lambda(k)$ is the smaller root of the equation (3.10). *Then* (1.1) *is oscillatory.*

4. General Oscillation Criteria

In this section we prove a general oscillation theorem for (1.1). We first mention two criteria which are immediate consequences of Lemmas 2.4 and 2.6, respectively.

Proposition 4.1. *Let* (1.12) *be fulfilled and the integral inequality* (2.8) *have no solution. Then the equation* (1.1) *is oscillatory.*

Proposition 4.2. *Let* (1.12) *be fulfilled and there exist a continuous function $\nu : R_+ \to R$ satisfying* (2.1) *and such that for any $T \ge \nu_{(-1)}(0)$ the integral inequality* (2.11) *has no solution. Then the equation* (1.1) *is oscillatory.*

Now we formulate our main result.

Theorem 4.1. *Let there exist continuous functions $\nu, \sigma, \delta : R_+ \to R$ such that σ, δ are nondecreasing,*

$$\begin{gathered} \nu(t) \le t, \ \tau(t) \le \delta(t) \le t, \ \ 0 < \sigma(t) \le \delta(t) \quad \text{for} \quad t \ge 0, \\ \nu(t), \sigma(t) \to +\infty \quad \text{as} \quad t \to +\infty, \end{gathered} \tag{4.1}$$

and for any $T \geq \tau_{(-1)}(0)$, *any positive solution* v *of* (2.8) *and any positive solution* w *of* (2.11) *the inequality*

$$\limsup_{t\to\infty}\left\{\int_{\delta(t)}^{t} p(s)\left(\tau(s)+\int_{T}^{\tau(s)}\xi\tau_{/\nu}(\xi)\,p(\xi)w(\xi)d\xi\right)\times\right.$$
$$\times\exp\left(\int_{\delta(s)}^{\delta(t)}\tau_T(\xi)p(\xi)v(\xi)d\xi\right)ds+$$
$$\left.+\left(\sigma(t)+\int_{T}^{\sigma(t)} s\tau_{/\nu}(\xi)\,p(\xi)w(\xi)d\xi\right)\int_{t}^{+\infty}\tau_{/\sigma}(\xi)\,p(\xi)\,d\xi\right\}>1 \tag{4.2}$$

holds, where τ_T *is defined by* (2.6) *and* $\tau_{/\nu},\tau_{/\sigma}$ *by* (2.3). *Then the equation* (1.1) *is oscillatory.*

Proof. First of all note that the condition (4.2) implies (1.12). Indeed, suppose that (1.12) is violated. Then by Lemmas 2.8 and 2.9 the integral equations (2.17) and (2.18) have solutions v_0 and w_0, respectively, such that $v_0(t)\leq M$ and $w_0(t)\leq M\,\tau(t)$ with some $M>1$. Using these inequalities along with the negation of (1.12), one can easily see that for $v\equiv v_0$ and $w\equiv w_0$ the left-hand side of (4.2) is zero. This proves that (1.12) holds.

Suppose now that, contrary to the assertion of the theorem, the equation (1.1) has a nonoscillatory solution $u:[t_0,+\infty[\to R$ which we may and will assume to be positive. Put $T=\tau_{(-1)}(t_0)$. By Lemma 2.1

$$u(\tau(t))\geq\tau_{/\sigma}(t)u(\sigma(t))\quad\text{for}\quad t\geq\max\{T,\sigma_{(-1)}(t_0)\}. \tag{4.3}$$

On the other hand, according to Lemmas 2.5 and 2.7 and because u' is nonincreasing, there exist positive solutions v and w of the integral inequalities (2.8) and (2.11), respectively, such that

$$u'(\tau(s))\geq u'(\delta(s))\geq E(v)(s,t)u'(\delta(t))\quad\text{for}\quad t\geq s\geq\tau_{(-2)}(T), \tag{4.4}$$

$$u(\tau(s))\geq F_\tau(w)(s)\,u'(\tau(s))\quad\text{for}\quad s\geq\tau_{(-1)}(T), \tag{4.5}$$

$$u(\sigma(t))\geq F_\sigma(w)(t)u'(\sigma(t))\ \geq F_\sigma(w)(t)u'(\delta(t))\ \text{for}\ \ t\geq\sigma_{-1}(T), \tag{4.6}$$

where for any $\mu:R_+\longrightarrow R$ we set

$$E(v)(s,t)\ =\ \exp\left(\int_{\delta(s)}^{\delta(t)}\tau_T(\xi)\,p(\xi)\,v(\xi)\,d\xi\right),$$
$$F_\mu(w)(t)\ =\ \left(\mu(t)+\int_{T}^{\mu(t)}\xi\tau_{/\nu}(\xi)\,p(\xi)\,w(\xi)\,d\xi\right).$$

Integrating (1.1) from $\delta(t)$ to $+\infty$ and taking into account (4.1) and (4.3)–(4.6) along with the nondecreasing character of u, σ, and δ, we get

$$u'(\delta(t)) \geq \int_{\delta(t)}^{t} p(s)\, u(\tau(s))\, ds + \int_{t}^{+\infty} p(s)\, u(\tau(s))\, ds \geq$$
$$\geq \int_{\delta(t)}^{t} p(s) F_\tau(w)(s) u'(\tau(s))\, ds + u(\sigma(t)) \int_{t}^{+\infty} \tau_{/\sigma}(s)\, p(s)\, ds \geq$$
$$\geq u'(\delta(t)) \left\{ \int_{\delta(t)}^{t} p(s) F_\tau(w)(s) E(v)(s,t) ds + F_\sigma(w)(t) \int_{t}^{+\infty} \tau_{/\sigma}(s) p(s)\, ds \right\}$$

for large t. But this contradicts (4.2). The proof is complete.

Remark 4.1. Propositions 4.1 and 4.2 can be considered as included in Theorem 4.1 by assuming formally that if there are no such v and w, then (4.2) is automatically fulfilled.

Theorem 4.1 and its corollaries below enable one to obtain effective sufficient conditions for the oscillation of (1.1) by means of a priori asymptotic lower estimates for v and w (or by means of establishing of nonexistence of v or w which in a way may be considered as the existence of a lower estimate identically equal to $+\infty$). We will derive nontrivial estimates of this type in Section 5.

Now we formulate some corollaries of the theorem. We begin with one which shows the joint effect of the delay and the second order nature of (1.1) in its simplest form.

Corollary 4.1. *Let τ be nondecreasing and*

$$\limsup_{t\to\infty} \left\{ \int_{\tau(t)}^{t} p(s)\tau(s) ds + \tau(t) \int_{t}^{+\infty} p(s)\, ds \right\} > 1.$$

Then the equation (1.1) *is oscillatory.*

Taking the first term in (4.2) with $\nu(t) \equiv t$ and using the obvious estimate $w(t) \geq t - T$, we obtain

Corollary 4.2. *Let there exist a nondecreasing function $\delta : R_+ \to R$ satisfying $\tau(t) \leq \delta(t) \leq t$ for $t \geq 0$ and such that for any solution v of* (2.8) *the inequality*

$$\limsup_{t\to\infty} \left\{ \int_{\delta(t)}^{t} p(s)\tau_0(s) \exp\left(\int_{\delta(s)}^{\delta(t)} \tau_0(\xi) p(\xi) v(\xi) d\xi \right) ds \right\} > 1$$

holds, where τ_0 is defined by (2.6). *Then the equation* (1.1) *is oscillatory.*

Corollary 4.2 shows the contribution of the delay to the oscillation of (1.1). As it has been pointed out in Section 3, some of (but not all) the results of that section could be derived from it.

Analogously, taking the second term in (4.2) with $\nu(t) \equiv t$ and using the estimate $w(t) \geq t - T$, we obtain

Corollary 4.3. *Let there exist a nondecreasing function* $\sigma : R_+ \to R$ *satisfying* $\sigma(t) \le \tau(t) \le t$ *for* $t \ge 0$, $\lim_{t\to+\infty} \sigma(t) = +\infty$ *and such that the inequality*

$$\limsup_{t\to\infty} \left\{ \left(\sigma(t) + \int_0^{\sigma(t)} s\tau(s)\, p(s) ds \right) \int_t^{+\infty} p(s)\, ds \right\} > 1$$

holds. Then the equation (1.1) *is oscillatory.*

In the case of ordinary differential equations Corollary 4.3 implies the following test.

Corollary 4.4. *If*

$$\limsup_{t\to\infty} \left\{ \left(t + \int_0^t s^2\, p(s) ds \right) \int_t^{+\infty} p(s)\, ds \right\} > 1, \tag{4.7}$$

then the equation

$$u''(t) + p(t)u(t) = 0 \tag{4.8}$$

is oscillatory.

Corollary 4.4 yields the following improvement of Hille's criteria (1.3) and (1.4) in the class of functions p satisfying

$$p(t) \ge \frac{c_0}{t^2} \quad \text{for large} \quad t. \tag{4.9}$$

Corollary 4.5. *Let* (4.9) *be fulfilled with* $c_0 \in]0, \frac{1}{4}]$ *and*

$$\limsup_{t\to+\infty} t \int_t^{+\infty} p(s) ds > \frac{1}{1 + c_0}.$$

Then (4.8) *is oscillatory.*

The condition (4.7) improves Hille's criteria even in the case where $c_0 = 0$. This is illustrated by the following

Example 4.1. Let the sequences of real numbers $\{a_k\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ be such that $a_k < b_k < a_{k+1}$ for $k = 1, 2, \dots$, $a_k \uparrow +\infty$ and $b_k \uparrow +\infty$ as $k \to \infty$, and

$$\lim_{k\to\infty} \frac{a_k}{b_k} = 0, \qquad \lim_{k\to\infty} \frac{b_k}{a_{k+1}} = 0$$

(for instance, we can take $a_k = 2^{k^2}$, $b_k = \frac{2^{k^2} + 2^{(k+1)^2}}{2}$). Let $\delta \in]0, \frac{3-\sqrt{5}}{2}[$ and $\varepsilon \in]0, 1[$ be such that $(1-\delta)(2-\delta)(1-\varepsilon) > 1$. Then for the function p defined by

$$p(t) = \begin{cases} \frac{1-\delta}{t^2} & \text{for } t \in]a_k, b_k[\\ 0 & \text{for } t \in]b_k, a_{k+1}[\end{cases}, \quad k = 1, 2, \dots,$$

both conditions (1.3) and (1.4) are violated while (4.7) is fulfilled. This means that Corollary 4.4 gives a positive answer to the question of oscillation of the equation (4.8) even in the case where both Hille criteria fail.

Indeed, we have

$$\limsup_{t\to+\infty} t\int_t^{+\infty} p(s)ds \le \limsup_{t\to+\infty} t\int_t^{+\infty} \frac{1-\delta}{s^2}ds = 1-\delta < 1$$

and

$$\liminf_{t\to+\infty} t\int_t^{+\infty} p(s)ds \le \lim_{k\to\infty} b_k \int_{b_k}^{+\infty} \frac{1-\delta}{s^2}ds = \lim_{k\to\infty} b_k \int_{a_{k+1}}^{+\infty} \frac{1-\delta}{s^2}ds =$$

$$= \lim_{k\to\infty} \frac{(1-\delta)b_k}{a_{k+1}} = 0.$$

On the other hand, denoting $a_k^* = a_k+\varepsilon(b_k-a_k)$, we have $a_k/a_k^* \to 0$ and $a_k^*/b_k \to \varepsilon$ as $k \to \infty$, so that

$$\limsup_{t\to+\infty} \left(t + \int_0^t s^2 p(s)ds\right) \int_t^{+\infty} p(s)\,ds \ge$$

$$\ge \limsup_{k\to\infty} \left(a_k^* + \int_{a_k}^{a_k^*} (1-\delta)\,ds\right) \int_{a_k^*}^{b_k} \frac{1-\delta}{s^2}\,ds \ge$$

$$\ge \limsup_{k\to\infty} \left(1 + (1-\delta)\left(1-\frac{a_k}{a_k^*}\right)\right)(1-\delta)\left(1-\frac{a_k^*}{b_k}\right) \ge$$

$$\ge (2-\delta)\,(1-\delta)\,(1-\varepsilon) > 1.$$

The following corollary will be used in Section 5 (we take $\delta(t) \equiv t$ and $\sigma = \nu$).

Corollary 4.6. *Let there exist a nondecreasing continuous function $\nu : R_+ \longrightarrow R$ such that $0 < \nu(t) \le \tau(t) \le t$, $\nu(t) \to +\infty$ as $t \to +\infty$ and for any $T \ge \nu_{(-1)}(0)$ and any solution w of (2.11) the inequality*

$$\limsup_{t\to\infty} \left(\nu(t) + \int_T^{\nu(t)} s\tau_{/\nu}(s)\,p(s)w(s)ds\right) \int_t^{+\infty} \tau_{/\nu}(s)\,p(s)\,ds > 1$$

holds, where $\tau_{/\nu}$ is defined by (2.3). Then the equation (1.1) is oscillatory.

Corollary 4.6, like Corollary 4.3, exhibits the role of the factors not depending on the presence of the delay. Next section is devoted to this topic.

5. Oscillations Due to the Second Order Nature of the Equation

In this section, using Corollary 4.6, we will derive oscillation criteria for (1.1) which are due to the second order nature of the equation. They generalize Hille's criterion (1.4) to delay equations.

Theorem 5.1. *Let $\alpha \in]0,1]$, $\tau(t) \ge \alpha t$ for large t and*

$$\liminf_{t\to+\infty} t\int_t^{+\infty} p(s)\,ds > c(\alpha), \tag{5.1}$$

where

$$c(\alpha) = \max\{\alpha^{\lambda-1}\lambda(1-\lambda) : 0 \le \lambda \le 1\}. \tag{5.2}$$

Then (1.1) is oscillatory.

Proof. Let $T \geq 0$, $\nu(t) \equiv \alpha t$, so that $\tau_{/\nu} \equiv 1$, and w be a solution of (2.11). By Corollary 4.6, it suffices to prove that the inequality

$$\limsup_{t\to+\infty}\left(\alpha t+\int_T^{\alpha t} sp(s)w(s)\,ds\right)\int_t^{+\infty}p(s)\,ds>1 \tag{5.3}$$

holds. This is the case if $\limsup\limits_{t\to+\infty}\alpha t\int_t^{+\infty}p(s)\,ds>1$, so we can suppose that

$$t\int_t^{+\infty}p(s)\,ds\leq 1/\alpha \quad \text{for large } t. \tag{5.4}$$

Put

$$\lambda_*=\liminf_{t\to+\infty}w(t)\left(\int_t^{+\infty}p(s)\,ds\right). \tag{5.5}$$

From (2.11) it is clear that $w(t)\geq \alpha t-T$, so $\lambda_*>\alpha c(\alpha)>0$. We claim that $\lambda_*>1$. Indeed, suppose to the contrary that $\lambda_*\in]0,1]$ and take $c_0\in\,]c(\alpha),\liminf\limits_{t\to+\infty} t\int_t^{+\infty}p(s)\,ds\,[$. By (5.1) and (5.5) for any $\lambda\in]0,\lambda_*[$ there is $t_0\geq T$ such that

$$w(t)\left(\int_t^{+\infty}p(s)\,ds\right)\geq\lambda,\quad t\int_t^{+\infty}p(s)\,ds\geq c_0 \quad \text{for } t\geq t_0. \tag{5.6}$$

Hence by (2.11) we have for $t\geq t_0/\alpha$

$$\begin{aligned} w(t)&\geq\int_{t_0}^{\alpha t}\exp\left\{\lambda\int_s^t p(\xi)\left(\int_\xi^{+\infty}p(\zeta)\,d\zeta\right)^{-1}d\xi\right\}ds= \\ &=\int_{t_0}^{\alpha t}\exp\left\{\lambda\ln\frac{\int_s^{+\infty}p(\zeta)\,d\zeta}{\int_t^{+\infty}p(\zeta)\,d\zeta}\right\}ds= \\ &=\left(\int_t^{+\infty}p(\zeta)\,d\zeta\right)^{-\lambda}\int_{t_0}^{\alpha t}s^{-\lambda}\left(s\int_s^{+\infty}p(\zeta)\,d\zeta\right)^{\lambda}ds\geq \\ &\geq\left(\int_t^{+\infty}p(\zeta)\,d\zeta\right)^{-\lambda}c_0^{\lambda}\,\frac{(\alpha t)^{1-\lambda}-t_0^{1-\lambda}}{1-\lambda}. \end{aligned} \tag{5.7}$$

Therefore by (5.6)

$$\begin{aligned} w(t)\int_t^{+\infty}p(\zeta)\,d\zeta&\geq\left(\int_t^{+\infty}p(\zeta)\,d\zeta\right)^{1-\lambda}\frac{(\alpha t)^{1-\lambda}c_0^{\lambda}}{1-\lambda}+o(1)= \\ &=\left(t\int_t^{+\infty}p(\zeta)\,d\zeta\right)^{1-\lambda}\frac{\alpha^{1-\lambda}c_0^{\lambda}}{1-\lambda}+o(1)\geq\frac{\alpha^{1-\lambda}c_0}{1-\lambda}+o(1). \end{aligned}$$

Passing here to lower limit, we get

$$\lambda_*\geq\frac{\alpha^{1-\lambda}c_0}{1-\lambda}.$$

Since $\lambda\in]0,\lambda_*]$ was arbitrary, we have

$$\alpha^{\lambda_*-1}\lambda_*(1-\lambda_*)\geq c_0>c(\alpha), \tag{5.8}$$

which contradicts (5.2). The obtained contradiction shows that $\lambda_* > 1$. Therefore (5.7) for any $\lambda \in]1, \lambda_*]$ yields

$$w(t)\int_t^{+\infty} p(\zeta)\,d\zeta \geq \frac{c_0^\lambda}{\lambda-1}\left(\int_t^{+\infty} p(\zeta)\,d\zeta\right)^{1-\lambda}\left[t_0^{1-\lambda}-(\alpha\,t)^{1-\lambda}\right]$$

which tends to $+\infty$ as $t \to +\infty$. This means that $\lambda_* = +\infty$ and so in the last inequality λ can be any number from $]1, +\infty[$. Rewrite this inequality as

$$w(t) \geq \frac{c_0^\lambda\, t^\lambda}{\lambda-1}\left(t\int_t^{+\infty} p(\zeta)\,d\zeta\right)^{-\lambda}\left[t_0^{1-\lambda}-(\alpha\,t)^{1-\lambda}\right].$$

Hence, in view of (5.4), it follows the existence of $M > 0$ and $t_1 \geq t_0$ such that

$$w(t) \geq M\,t^\lambda \quad \text{for} \quad t \geq t_1,$$

i.e., for any $\lambda > 1$

$$w(t) \geq t^\lambda \quad \text{for large} \quad t. \tag{5.9}$$

Using (5.9) for $\lambda = 2$ along with (5.2) and (5.4), and integrating by parts, we get for large t

$$\begin{aligned}\int_T^{\alpha\,t} sp(s)w(s)\,ds &\geq \int_{t^{1/2}}^{\alpha\,t} s^3 p(s)\,ds \geq \\ &\geq -t\int_{t^{1/2}}^{\alpha\,t} s\,d\left(\int_s^{+\infty} p(\xi)\,d\xi\right) = \\ = t\left(t^{1/2}\int_{t^{1/2}}^{+\infty} p(\xi)\,d\xi - \alpha t\int_{\alpha\,t}^{+\infty} p(\xi)\,d\xi + \int_{t^{1/2}}^{\alpha\,t}\left(\int_s^{+\infty} p(\xi)\,d\xi\right)ds\right) &\geq \\ \geq t\left(-\frac{1}{\alpha} + \int_{t^{1/2}}^{\alpha\,t}\frac{c(\alpha)}{s}ds\right) = t\left(-\frac{1}{\alpha} + c(\alpha)\ln\alpha + \frac{c(\alpha)}{2}\ln t\right).\end{aligned} \tag{5.10}$$

Hence, in view of (5.1), we have (5.3). The proof is complete.

Remark 5.1. The constant $c(\alpha)$ is best possible in the sense that in (5.1) the strict inequality cannot be replaced by the nonstrict one without affecting the validity of the theorem. Indeed, denoting by λ_0 the point where the maximum in (5.2) is attained, we can see that the function $u(t) \equiv t^{1-\lambda_0}$ is a nonoscillatory solution of the equation $u''(t) + (c(\alpha)/t^2)u(\alpha t) = 0$.

Remark 5.2. We have $\alpha c(\alpha) = \max\{\alpha^\lambda\lambda(1-\lambda) : 0 \leq \lambda \leq 1\} < \max\{\lambda(1-\lambda) : 0 \leq \lambda \leq 1\} = 1/4$ for $0 < \alpha < 1$. Therefore for any $\alpha \in]0, 1[$ Theorem 5.1 improves the result of Wong (1.5).

Remark 5.3. Using Corollary 4.6 with $\nu(t) \equiv t$, we could analogously to Theorem 5.1 derive the criterion (1.7).

Now consider the case where (5.1) is violated.

Theorem 5.2. *Let* $\alpha \in]0,1], \tau(t) \geq \alpha t$ *for large* t,

$$\liminf_{t\to+\infty} t \int_t^{+\infty} p(s)\,ds = c_0 \in]0, c(\alpha)] \tag{5.11}$$

and

$$\limsup_{t\to+\infty} \left(\alpha t + \lambda_0 \int_0^{\alpha t} sp(s) \left(\int_t^{+\infty} p(\xi)\,d\xi\right)^{-1} ds\right) \int_t^{+\infty} p(s)\,ds > 1, \tag{5.12}$$

where $c(\alpha)$ *is defined by* (5.2) *and* λ_0 *is the smaller root of the equation*

$$\alpha^{\lambda-1}\lambda(1-\lambda) = c_0. \tag{5.13}$$

Then (1.1) *is oscillatory.*

Proof. In view of (5.11)–(5.13) one can choose $c^* \in]0, c_0[$ close enough to c_0, $\varepsilon > 0$ small enough and $t_0 \geq 0$ large enough for the inequalities

$$t \int_t^{+\infty} p(s)\,ds \geq c^* \quad \text{for} \quad t \geq t_0$$

and

$$\limsup_{t\to+\infty} \left(\alpha t + (\lambda^* - \varepsilon) \int_0^{\alpha t} sp(s) \left(\int_t^{+\infty} p(\xi)\,d\xi\right)^{-1} ds\right) \int_t^{+\infty} p(s)\,ds > 1 \tag{5.14}$$

to hold, where λ^* is the smaller root of $\alpha^{\lambda-1}\lambda(1-\lambda) = c^*$.

Let w be a solution of (2.11) with $\nu(t) \equiv \alpha t$. Defining λ_* by (5.5) and acting as in deriving the inequality (5.8), we get

$$\alpha^{\lambda_*-1}\lambda_*(1-\lambda_*) \geq c^*,$$

whence we get $\lambda_* \geq \lambda^*$. This means that

$$w(t) \geq (\lambda^* - \varepsilon) \left(\int_t^{+\infty} p(\xi)\,d\xi\right)^{-1} \quad \text{for large } t.$$

Therefore (5.14) and Corollary 4.6 imply that the equation (1.1) is oscillatory. The proof is complete.

In the class of the functions p satisfying

$$p(t) \geq \frac{c_0}{t^2} \quad \text{for large} \quad t, \tag{5.15}$$

we can get the following result which is similar to Theorem 3.4 in the sense that it connects the upper and lower limits of the same expression.

Theorem 5.3. *Let* $\tau(t) \geq \alpha t$ *for large* t *and* (5.15) *be fulfilled, where* $\alpha \in]0,1]$, $c_0 \in]0, c(\alpha)]$ *and* $c(\alpha)$ *is defined by* (5.2). *Let, moreover,*

$$\limsup_{t\to+\infty} t \int_t^{+\infty} p(s)\,ds > \frac{1}{\alpha(1+\lambda_0)}, \tag{5.16}$$

where λ_0 *is the smaller root of* (5.13). *Then* (1.1) *is oscillatory.*

Proof. Let $T \geq 0$ and w be a solution of (2.11) with $\nu(t) \equiv \alpha t$. Denote the left-hand side of (5.16) by p^*. By (5.16) there is a sufficiently small $\varepsilon > 0$ such that $p^*\alpha\,(1+\lambda_0-\varepsilon) > 1$. According to Corollary 4.6 it suffices to prove that

$$w(t) \geq \frac{\lambda_0 - \varepsilon}{c_0} t \quad \text{for large } t. \tag{5.17}$$

Denote $\beta_0 = \liminf_{t\to+\infty} w(t)/t$ (from (2.11) it follows that $\beta \geq \alpha > 0$), so for any $\beta \in]0, \beta_0[$ there is $t_0 \geq T$ such that $w(t) \geq \beta t$ for $t \geq t_0$. Suppose first that $\beta c_0 > 1$. Then (2.11) yields

$$\begin{aligned} w(t) &\geq \int_{t_0}^{\alpha t} \exp\left\{\int_s^t \frac{\beta c_0}{\xi} d\xi\right\} ds \geq \int_{t_0}^{\alpha t} \left(\frac{t}{s}\right)^{\beta c_0} ds = \\ &= \frac{\alpha^{1-\beta c_0} t}{\beta c_0 - 1}\left(\left(\frac{t_0}{\alpha t}\right)^{1-\beta c_0} - 1\right) \quad \text{for} \quad t \geq t_0/\alpha. \end{aligned} \tag{5.18}$$

So (5.17) is fulfilled. Analogously, if $\beta c_0 = 1$, then $w(t) \geq t\ln(\alpha t/t_0)$ for large t, and again (5.17) holds. Finally, let $\beta c_0 < 1$. Then, from (5.18), we get

$$\frac{w(t)}{t} \geq \frac{\alpha^{1-\beta c_0}}{1-\beta c_0}\left(1 - \left(\frac{t_0}{\alpha t}\right)^{1-\beta c_0}\right) \quad \text{for large} \quad t.$$

Since $\beta \in]0, \beta_0[$ is arbitrary, passing to lower limit we obtain that $\lambda = \beta_0 c_0$ satisfies $\alpha^{\lambda-1}\lambda(1-\lambda) \geq c_0$. Hence $\beta_0 c_0 \geq \lambda_0$ which means that (5.17) is fulfilled. The proof is complete.

Finally we consider the case where the delay, roughly speaking, is like t^α.

Theorem 5.4. *Let $\alpha \in]0, 1[$ and $\liminf_{t\to+\infty} \tau(t)t^{-\alpha} > 0$. Then the condition*

$$\liminf_{t\to+\infty} t^\alpha \int_t^{+\infty} p(s)\,ds > 0$$

is sufficient for (1.1) *to be oscillatory.*

Proof. The proof is quite analogous to that of Theorem 5.1, so it will be only sketched. Let $T \geq 0$, $\gamma > 0$ be such that $\tau(t) \geq \gamma t^\alpha$ for large t and w be a solution of (2.11) with $\nu(t) \equiv \gamma t^\alpha$. Define λ_* by (5.5) and suppose first that $\alpha\lambda_* \leq 1$. Let $\lambda < \lambda_*$ and $\beta > 0$ be such that $t^\alpha \int_t^{+\infty} p(s)\,ds \geq \beta$ for large t. Proceeding as in deriving (5.7), we obtain

$$\begin{aligned} w(t)\int_t^{+\infty} p(\zeta)\,d\zeta &\geq \frac{\beta^\lambda \gamma^{1-\alpha\lambda}}{1-\alpha\lambda}\left(\int_t^{+\infty} p(\zeta)\,d\zeta\right)^{1-\lambda} t^{\alpha(1-\alpha\lambda)}\,[1+o(1)] \geq \\ \geq \frac{\beta^\lambda\gamma^{1-\alpha\lambda}}{1-\alpha\lambda}\left(t^\alpha \int_t^{+\infty} p(\zeta)\,d\zeta\right)^{1-\lambda} & t^{\alpha\lambda(1-\alpha)}\,[1+o(1)] \geq Ct^{\alpha\lambda(1-\alpha)}\,[1+o(1)] \to +\infty \end{aligned}$$

as $t \to +\infty$, where $C > 0$ is a constant. We were able to write the last inequality since like in (5.7) we can assume that $t^\alpha \int_t^{+\infty} p(s)\,ds \leq 1$ for large t and therefore the $(1-\lambda)$-th power can be estimated from below independently of whether $\lambda > 1$

or $\lambda \le 1$. Thus we have $\lambda_* = +\infty$ which contradicts our assumption that $\alpha\lambda_* \le 1$. Thus $\alpha\lambda_* > 1$ and we can take $\lambda \in]1/\alpha, \lambda_*[$ to get

$$w(t) \int_t^{+\infty} p(\zeta)\, d\zeta \ge \frac{\beta^\lambda \gamma^{1-\alpha\lambda}}{\alpha\lambda - 1} \left(\int_t^{+\infty} p(\zeta)\, d\zeta \right)^{1-\lambda} [C_0 + o(1)] \to +\infty$$

as $t \to +\infty$, where $C_0 > 0$ is a constant. Therefore $\lambda_* = +\infty$. Hence as in the proof of Theorem 5.1 we conclude that (5.9) holds for any $\lambda > 0$. Using this inequality with $\lambda = 2$ and writing down the chain of inequalities analogous to (5.9) (instead of $t^{1/2}$ one has to take $t^{\alpha/2}$), we can ascertain that the conditions of Corolllary 4.6 are fulfilled with $\nu(t) \equiv \gamma t^\alpha$. The proof is complete.

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EXTREMITIES OF INFINITE PRODUCTS AND THE HÖLDER'S INEQUALITY

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ABSTRACT. We use an extension of the Hölder's inequality to solve some min-max problems concerning infinite products.

1. INTRODUCTION

The purpose of this note is to show how an extension of the Hölder's inequality can be used to obtain extreme points of functionals defined by infinite products. More specifically consider the following problems:

Problem 1: Let D^+ denote the positive orthant of the unit ball in l^1 and consider a fixed $b := (b_n) \in D^+$ with norm $\|b\|_1 = 1$. We are interested in maximizing the infinite product

$$\prod {a_n}^{b_n},$$

over all sequences $(a_n) \in D^+$.

Problem 2: Consider a finite positive measure space X, m and a complex valued function $f \in L^\infty(X,m)$. Then $f \in L^p(X,m)$, for all $p \in [1, +\infty]$. We denote by $\|f\|_p$ the usual semi-norm of f as an element of the space $L^p(X,m)$. If $\sum a_n$ is a convergent series of positive real numbers, we are interested in minimizing the quantity

$$\prod \|f^{a_n}\|_{p_n},$$

over all sequences (p_n) of positive real numbers with $\sum \frac{1}{p_n} = 1$.

It is well known that the previous problems can be solved by seeking local minimum or maximum of the corresponding functionals. Here we shall give answers to these problems by extending the well known Hölder's inequality to infinite products of functions. Also by this result many useful inequalities concerning series and infinite products can be obtained.

Hölder's inequality holds for any finite set of factors, (see, e.g., [1, pp. 231-232], or [2, pp. 63-64], or [3, p. 210], or [4, p. 68], or [5, p. 86]) and it is met in the following form:

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Let $f_1, f_2, ..., f_n$ be Lebesgue measurable functions. If $0 < p_i \leq +\infty, i = 0, 1, ..., n$ are such that $p_0^{-1} = p_1{}^{-1} + p_2{}^{-1} + ... + p_n{}^{-1}$, and if $f_i \in L^{p_i}$, then the product $f_1 ... f_n$ is an element of the space L^{p_0} and it holds

$$\|f_1.f_2...f_n\|_{p_0} \leq \|f_1\|_{p_1}.\|f_2\|_{p_2}...\|f_n\|_{p_n}. \tag{1.1}$$

The same proof works to show that inequality (1.1) is true for $L^p(X, m)$, where X, m is any (positive) measure space. It is the purpose of this note to show how inequality (1.1) can be extended to (infinite) sequences of functions.

2. The main result

Let X be a nonempty set and let m be a positive measure on X.

Theorem. *Let p_0 be a positive real number and let (p_i) be a sequence such that $1 \leq p_i \leq +\infty$, $i = 1, 2, ...$ and*

$$p_0{}^{-1} = \sum p_i{}^{-1}.$$

(Notice that $\frac{1}{+\infty}$ is taken to be 0.) Assume also that for each index i a function f_i is given in the space $L^{p_i}(X, m)$. If the infinite product Πf_i converges a.e. on X to a certain f, then f belongs to $L^{p_0}(X, m)$ and it satisfies the inequality

$$\|f\|_{p_0} \leq \prod \|f_i\|_{p_i}. \tag{2.1}$$

Proof. Assume first that $p_0 = 1$ and $m(X) = 1$.

Let B represent the right side of inequality (2.1). If $B = +\infty$, then we have nothing to show. So, assume that $B < +\infty$.

We suppose that $B > 0$. Then $lim\|f_n\|_{p_n} = 1$. For each n let q_n be a real number defined by

$$\frac{1}{q_{n+1}} = 1 - \sum_{j=1}^{n} \frac{1}{p_j}.$$

Then $limq_n = +\infty$ and, moreover, $p_{n+1} \geq q_{n+1}$, for all n. Therefore each element h of the space $L^{p_{n+1}}(X, m)$ is also an element of $L^{q_{n+1}}(X, m)$ and it holds

$$\|h\|_{q_{n+1}} \leq \|h\|_{p_{n+1}}. \tag{2.2}$$

Let $\epsilon > 0$ be a fixed real number. Then there is an index n_0 such that

$$\prod_{i=1}^{n} \|f_i\|_{p_i} < Be^{\epsilon}, \tag{2.3}$$

for all $n \geq n_0$. Fix such an index n and observe that because of (1.1) and (2.3) it holds

$$\int |f_1.f_2...f_n.f_{n+1}|dm \leq \prod_{i=1}^{n}[\int |f_i|^{p_i}dm]^{\frac{1}{p_i}}[\int |f_{n+1}|^{q_{n+1}}dm]^{\frac{1}{q_{n+1}}}$$

$$\leq Be^{\epsilon}\|f_{n+1}\|_{q_{n+1}}$$

Therefore taking into account relation (2.2) we get

$$\int |f_1.f_2...f_n.f_{n+1}|dm \leq Be^{\epsilon}.\|f_{n+1}\|_{p_{n+1}}.$$

Applying Fatou's lemma and keeping in mind that the number ϵ is arbitrary, we get

$$\|f\|_1 = \|\prod f_i\|_1 \leq \prod \|f_i\|_{p_i}. \qquad (2.4)$$

in the case of $B > 0$.

If $B = 0$, then (2.4) also holds. To see this we follow the same procedure as above with the factor Be^{ϵ} being replaced by the number ϵ and taking into account that $\|f_{n+1}\|_{p_{n+1}} < 1$, eventually for all indices n.

If X has measure $m(X) =: M > 0$, we can apply the previous inequality with respect to the measure $\frac{m}{M}$ to see that (2.4) also holds.

If the space X is σ-finite, it can be written as the union of an increasing sequence of sets X_j with finite measure. Then (2.4) is valid over all subspaces X_j, namely, for $f_i \in L^{p_i}(X,m)$ it holds

$$\int_{X_j} |f|dm \leq \prod[\int_{X_j} |f_i|^{p_i}dm]^{\frac{1}{p_i}},$$

or

$$\int |f.\chi_{X_j}|dm \leq \prod[\int |f_i.\chi_{X_j}|^{p_i}dm]^{\frac{1}{p_i}},$$

for all indices j. Applying the monotone convergence theorem, we easily get (2.4).

Finally, assume that p_0 is not necessarily equal to 1. Then, considering the numbers $\frac{p_i}{p_0}$ and applying (2.4) to $|f_i|^{p_0}$ we obtain inequality (2.1). The proof of the theorem is complete. □

Remark.

If m is the discrete measure, then inequality (2.1) takes the form

$$[\sum_i[\prod_n |c_{i,n}|]^{p_0}]^{\frac{1}{p_0}} \leq \prod_n[\sum_i |c_{i,n}|^{p_n}]^{\frac{1}{p_n}}$$

for any sequence $(c_{i,n})$.

3. The answers to the problems and some inequalities

Assume that $a := (a_n)$ and $b := (b_n)$ are two convergent sequences of nonnegative real numbers. Applying the previous theorem on the corresponding sequences of functions we get the following inequalities:

1. $X := (-\infty, 0]$, $f_n(s) := exp(a_n s)$ and $p_n := \frac{\|b\|_1}{b_n}$, or $X := \mathbb{R}$, $f_n(s) := exp(\frac{-s^2 a_j}{\|a\|_1})$ and $p_n := \frac{\|b\|_1}{b_n}$, or $X := (0, +\infty)$, $f_n(s) := s^{b_n}e^{-sa_n}$ and $p_n := \frac{\|a\|_1}{a_n}$. Then we get

$$[\sum b_n]^{\sum b_n} \prod {a_m}^{b_m} \leq [\sum a_n]^{\sum b_n} \prod {b_m}^{b_m}.$$

Thus, the maximum of the quantity $\prod {a_n}^{b_n}$, over all (a_n) in the positive orthant D^+ of the unit disc in the space l_1, is attained by the sequence $(\frac{b_n}{\|b\|_1})$, where $\|b\|_1$ is the norm of the sequence (b_n) as an element of the space l_1, i.e. the limit of the series $\sum b_n$. This solves our first problem. For example one can easily derive that

$$max\{\prod a_n^{\frac{1}{2^n}} : (a_n) \in D^+\} = \frac{1}{4}.$$

2. $X := [0,1]$, $f_n(s) := s^{ta_n}$ and $p_n := \frac{\|b\|_1}{b_n}$. Then for any real number t with $t\|a\|_1 + 1 > 0$, we get

$$\prod (ta_n\|b\|_1 + b_n)^{b_n} \leq (t\|a\|_1 + 1)^{\|b\|_1} \prod b_n^{b_n}.$$

3. $X := [0,1]$, $t > -1$, $f_n(s) := s^{\frac{ta_n}{\|a\|_1}}$ and $p_n := \frac{\|b\|_1}{b_n}$. Then we obtain

$$\prod (ta_n\|b\|_1 + b_n\|a\|_1)^{b_n} \leq [(t+1)\|a\|_1]^{\|b\|_1} \prod b_n^{b_n}.$$

4. Applying the above main result we conclude that given a sequence (p_n) of positive real numbers, with $\sum {p_n}^{-1} = 1$, it holds $\int |f|^a dm \leq \prod \|f^{a_n}\|_{p_n}$, where a is the limit of the series $\sum a_n$. Equality holds for $p_n := {a_n}^{-1}$ for all n. Therefore the minimum of the infinite product $\prod \|f^{a_n}\|_{p_n}$ over all such sequences (p_n) is equal to $\||f|^a\|_1$ and it is attained by the sequence $p_n := a{a_n}^{-1}$. This gives the answer to the second problem in the begining of this note.

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ON AN INTEGRAL FUNCTIONAL INEQUALITY

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ABSTRACT. In this paper, we establish upper bounds for the solutions of some functional integral inequalities.

1. INTRODUCTION

In a recent paper Pachpate [1] has obtained upper bounds for the solutions of the following integral inequalities

$$\text{(A)} \qquad x^2(t) \le c^2 + 2\int_0^t \left[f(s)x(\sigma(s))W(x(\sigma(s))) + h(s)x(\sigma(s))\right] ds,$$

$$\text{(B)} \quad x^2(t) \le c^2 + 2\int_0^t \left[f(s)x(\sigma(s))\left(\int_0^s g(\tau)W(x(\sigma(\tau)))d\tau\right) + h(s)x(\sigma(s))\right] ds,$$

(L)

$$x^2(t) \le c^2 + 2\int_0^t \left[f(s)x(\sigma(s))\left(\int_0^s g(\tau)W(log x(\sigma(\tau)))d\tau\right) + h(s)x(\sigma(s))\right] ds,$$

for $t \in [0, \infty)$, with the conditions

$$x(t) = \psi(t) \le c, t \in [\min_{t\in\mathbb{R}^+} \sigma(t), 0],$$

and

$$\sigma \in C([0,\infty), \mathbb{R}), \quad \text{with} \quad \sigma(t) \le t, t \in [0, \infty).$$

The purpose of this note is to obtain upper bounds for the solutions of more general integral inequalities of the following form

$$(A_1) \qquad x^n(t) \le c^n + n\int_0^t \left[f(s)x^m(\sigma_1(s))W(x^r(\sigma_2(s))) + h(s)x^l(\sigma_3(s))\right] ds,$$

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$$n > r > 0, \quad 0 < m, l \leq n - r,$$

(B_1)

$$x^n(t) \leq c^n + n \int_0^t \left[f(s)x^m(\sigma_1(s)) \left(\int_0^s g(\tau)W(x^r(\sigma_2(\tau)))d\tau \right) + h(s)x^l(\sigma_3(s)) \right] ds,$$

$$n > r > 0, \quad 0 < m, l \leq n - r,$$

(L_1)

$$x^n(t) \leq c^n + n \int_0^t \left[f(s)x^m(\sigma_1(s)) \left(\int_0^s g(\tau)W(log x(\sigma_2(\tau)))d\tau \right) + h(s)x^l(\sigma_3(s)) \right] ds,$$

$$n > 0, \quad 0 < m, l \leq n,$$

where $t \in [0, \infty)$.

Also, in all above cases we suppose that

(C) $x(t) = \psi(t) \leq c, t \in [a, 0]$, where ψ is a given real function defined on $[a, 0]$ and $a = \min_{i=1,2,3} \{\min \sigma_i(t) : t \in [0, \infty)\}$.

2. Main Result

For our convenience we list bellow the assumptions we will use in the next theorem.

- (H_1) $\sigma_i \in C([0, \infty), R)$, with $\sigma_i(t) \leq t, \quad t \in [0, \infty), i = 1, 2, 3.$
- (H_2) $f \in C([0, \infty), [0, \infty))$.
- (H_3) $h \in C([0, \infty), [0, \infty))$.
- (H_4) $g \in C([0, \infty), [0, \infty))$.
- (H_5) $x \in C([a, \infty), [x_0, \infty)), x_0 \geq 0, c \geq 1$.
- (H_6) $x \in C([a, \infty), [x_0, \infty)), x_0 \geq 1, c \geq 1$.
- (H_7) $W \in C([0, \infty), [0, \infty))$ is nondecreasing, $W(x) \geq 0$, $x > x_0$ and $W(x_0) = 0$.

Theorem. *(i) Inequality (A_1) with (C) and assumptions $(H_i), i = 1, 2, 3, 5, 7$ imply*

(1) $$x(t) \leq \left[G^{-1} \left[G \left(c^r + r \int_0^t h(s)ds \right) + r \int_0^t f(s)ds \right] \right]^{\frac{1}{r}}, 0 \leq t \leq \beta_1.$$

(ii) Inequality (B_1) with (C) and assumptions $(H_i), i = 1, 2, 3, 4, 5, 7$ imply

(2)

$$x(t) \leq \left[G^{-1} \left[G \left(c^r + r \int_0^t h(s)ds \right) + r \int_0^t f(s) \left(\int_0^s g(\tau)d\tau \right) ds \right] \right]^{\frac{1}{r}}, 0 \leq t \leq \beta_2.$$

(iii) Inequality (L_1) with (C) and assumptions $(H_i), i = 1, 2, 3, 4, 6, 7$ imply

(3)

$$x(t) \leq exp\left[G^{-1}\left[G\left(logc + \int_0^t h(s)ds\right) + \int_0^t f(s)\left(\int_0^s g(\tau)d\tau\right)ds\right]\right], 0 \leq t \leq \beta_3,$$

where

$$G(u) = \int_0^u \frac{ds}{W(s)}, u \geq u_0 > x_0,$$

G^{-1} is the inverse of G and the numbers $\beta_i, i = 1, 2, 3$ are choosen so that the quantities in the square bracets in (1), (2), (3) are in the range of G.

Proof. In the following we give in details the proof of (ii) and (iii). The proof of the assertion (i) can be done easily following the proof of (ii).

(ii) We define a function $u\colon [0,\infty) \to [0,\infty)$, by

$$u^n(t) = (c+\varepsilon)^n + n\int_0^t \left[f(s)x^m(\sigma_1(s))\left(\int_0^s g(\tau)W(x^r(\sigma_2(\tau)))d\tau\right) + h(s)x^l(\sigma_3(s))\right]ds,$$

where $\varepsilon > 0$, is an arbitrary constant. Then we have

$$u(0) = c + \varepsilon$$

and

(4)

$$nu^{n-1}(t)u'(t) = nf(t)x^m(\sigma_1(t))\int_0^t g(\tau)W(x^r(\sigma_2(\tau))d\tau + h(t)x^l(\sigma_3(t)), t \in [0,\infty).$$

Now, if $0 \leq \sigma_1(t) \leq t$, we have

$$\begin{aligned} x^n(\sigma_1(t)) &< u^n(\sigma_1(t)) \\ &= (c+\varepsilon)^n \\ &\quad + n\int_0^{\sigma_1(t)} \left[f(s)x^m(\sigma_1(s))\left(\int_0^s g(\tau)W(x^r(\sigma_2(\tau)))d\tau\right) + h(s)x^l(\sigma_3(s))\right]ds \\ &\leq (c+\varepsilon)^n \\ &\quad + n\int_0^t \left[f(s)x^m(\sigma_1(s))\left(\int_0^s g(\tau)W(x^r(\sigma_2(\tau)))d\tau\right) + h(s)x^l(\sigma_3(s))\right]ds \\ &= u^n(t). \end{aligned}$$

Also, if $a \leq \sigma_1(t) \leq 0$, we have

$$x(\sigma_1(t)) = \psi(\sigma_1(t)) < c + \varepsilon < u(t).$$

Thus, in any case we have

$$x(\sigma_1(t)) < u(t), t \in [0,\infty).$$

Similarly, we have also

$$x(\sigma_i(t)) < u(t), t \in [0, \infty), i = 2, 3.$$

By (4), since $0 < m, l \leq n - r$, we have

$$u^{n-1}(t)u'(t) < f(t)u^{n-r}(t) \int_0^t g(\tau)W(u^r(\tau))d\tau + u^{n-r}(t)h(t), t \in [0, \infty).$$

or

$$u^{r-1}(t)u'(t) < f(t) \int_0^t g(\tau)W(u^r)(\tau)d\tau + h(t), t \in [0, \infty).$$

Integrating both sides from 0 to t we have

$$u^r(t) < p(t) + r \int_0^t f(s) \left(\int_0^s g(\tau)W(u^r(\tau))d\tau \right) ds, t \in [0, \infty),$$

where

$$p(t) = (c + \varepsilon)^r + r \int_0^t h(s)ds, t \in [0, \infty).$$

For an arbritrary $T \in [0, \infty)$ we have

$$u^r(t) < p(T) + r \int_0^t f(s) \left(\int_0^s g(\tau)W(u^r(\tau))d\tau \right) ds, t \in [0, \infty).$$

We set

$$v(t) = p(T) + r \int_0^t f(s) \left(\int_0^s g(\tau)W(u^r(\tau))d\tau \right) ds, t \in [0, \infty).$$

Since $u^r(t) < v(t), \quad t \in [0, T]$ and W is nondecreasing, we have

$$v'(t) \leq rf(t)W(v(t)) \int_0^t g(\tau)d\tau, t \in [0, T].$$

Thus

$$\frac{d}{dt}G(v(t)) \leq rf(t) \int_0^t g(\tau)d\tau, t \in [0, T].$$

Integrating both sides from 0 to T, we have

$$G(v(T)) \leq G(p(T)) + r \int_0^T f(s)(\int_0^s g(\tau)d\tau)ds.$$

Hence

$$v(T) \leq G^{-1} \left[G(p(T)) + r \int_0^T f(s) \left(\int_0^s g(\tau)d\tau \right) ds \right].$$

Since T is arbitrary and $x(t) < u(t), \quad t \in [0,\infty)$, the result is obvious by letting $\varepsilon \to 0$.

(iii) We define a function u: $[0,\infty) \to [0,\infty)$, by

$$u^n(t) = (c+\varepsilon)^n + n\int_0^t \left[f(s)x^m(\sigma_1(s)) \left(\int_0^s g(\tau)W(logx(\sigma_2(\tau)))d\tau \right) + h(s)x^l(\sigma_3(s)) \right] ds,$$

where $\varepsilon > 0$, is an arbitrary constant. Then

$$u(0) = c + \varepsilon$$

and for every $t \in [0,\infty)$ we have

$$nu^{n-1}(t)u'(t) = nf(t)x^m(\sigma_1(t)) \int_0^t g(\tau)W(logx(\sigma_2(\tau)))d\tau + h(t)x^l(\sigma_3(t)). \tag{5}$$

As in the proof of step (i) and, since $0 < m, l \leq n$, we can prove that

$$x^m(\sigma_i(t)) < u^n(t) \quad \text{and} \quad x^l(\sigma_i(t)) < u^n(t), \quad t \in [0,\infty), i = 1,2,3.$$

Hence, by (5) we have

$$u^{n-1}(t)u'(t) < f(t)u^n(t) \int_0^t g(\tau)W(logu(\tau))d\tau + h(t)u^n(t), t \in [0,\infty).$$

or

$$\frac{u'(t)}{u(t)} \leq f(t) \int_0^t g(\tau)W(logu(\tau))d\tau + h(t), t \in [0,\infty).$$

Integrating both sides from 0 to t, we have

$$logu(t) \leq \hat{p}(t) + \int_0^t f(s)(\int_0^s g(\tau)W(logu(\tau))d\tau)ds, t \in [0,\infty),$$

where

$$\hat{p}(t) = log(c+\varepsilon) + \int_0^t h(s)ds.$$

We omit the rest of the proof since it is similar to that in the above step (i). □

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ON THE SYMMETRIES AND SIMILARITY SOLUTIONS OF ONE-DIMENSIONAL, NON-LINEAR THERMOELASTICITY

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Abstract

The homogeneous, one–dimensional, non–linear thermoelasticity is studied from the point of view of symmetries and similarity solutions. Special cases of free energy function and conductivity function are considered and the corresponding admitted symmetry group of transformations are derived. Also, the similarity solutions, if any, for each symmetry group are provided. Finally, the whole procedure is checked by means of obtaining the reduction of the system of partial differential equations to a system of ordinary differential equations by the insertion of the similarity solutions into them.

1 Introduction

This paper is the second part of a work concerning the symmetries of non-linear, one-dimensional, dynamical thermoelasticity. In the first part [1] the general non-homogeneous problem was concidered. In [2] some preliminary results concerning the homogeneous case are presented. The present paper intends to exhaust the homogeneous case.

By the term "non-linear", we mean non-linearity coming into the system through constitutive relations i.e., assuming a general non-linear (actually more than quadratic) free energy function. On the other hand, this is not the real full non-linear thermoelasticity because the linear relation between heat conduction and temperature field i.e., the well-known Fourier law is considered.

The concept of symmetry of a differential equation has been introduced by Sophus Lie one hundred years ago. To find out the symmetries of a differential equation means to find out the continuous group of transformations (actually they are Lie groups) under which the differential equation is invariant. Having such symmetries, one can obtain new solutions from an existing one and the so-called group-invariant solutions [3]; the well-known similarity (or self-similar) solutions is nothing but a special case of group-invariant solutions corresponding to the scaling group. Generally speaking, the more symmetries of a differential equation we know the more we know about the differential equation itself.

The fundamental ideas of S. Lie can be fruitfully coupled with concepts coming from exterior calculus [4]. According to this the main idea is based on Cartan's work by which one can obtain a geometric description of a partial differential equation in terms of closed ideals of exterior differential forms. Next, one has to find out the so-called isovector fields which in turn are defined to be the vector fields on the space of dependent and indepedent variables, over which the Lie derivatives of Cartan's exterior differential forms remain invariant. These are nothing more but the infinitesimal generators [3] of the Lie group of transformations.

Most of the researchers of the area use the so–called determining equations i.e., an overdetermined system of linear PDEs which govern the components of the isovector field, to obtain the symmetries of a differential equation. This procedure to obtain symmetries relies directly on Lie groups theory applied to the particular case of transformations group. We refer to the books of Ibragimov [5], Olver [3] and Bluman and Cole [6] for further information for the interested reader.

In Sect. 2, we summarize some results derived in [1] which are useful to the present paper. In Sect. 3, we give the isovector field for the homogeneous thermoelasticity and in Sect. 4, we examine special cases of symmetries corresponding to various cases of free energy and conductivity functions. Finally, in Sect. 5, we provide the similarity solutions arising from every non–trivial symmetry.

2 Equations of Thermoelasticity and Some Previous Results

In the first part of this work [1] we examined the system of thermoelasticity equations:

$$\frac{\partial}{\partial X}(\frac{\partial F}{\partial p})-\frac{\partial}{\partial t}(\rho_0(X)v)=0,$$
$$\frac{\partial}{\partial X}(k(X)\frac{\partial \theta}{\partial X})+\frac{\partial}{\partial t}(\theta\frac{\partial F}{\partial \theta})-\frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial t}=0, \tag{1}$$

where $x(X,t)$ is the motion of the body, $p = gradx$ is the gradient of deformation, $v = \frac{\partial x}{\partial t}$ is the velocity, $\theta(X,t)$ is the absolute temperature field, $\rho(X)$ is the mass density, $k = k(X)$ is the conductivity function and $F = F(X,p,\theta)$ is the free energy function. By X is denoted the material coordinate and by t the time. The symmetries of this system are given by the components of the isovector field given by the equations [1]

$$\begin{aligned}
&\omega^1(X) = aX + c_1,\\
&\omega^2(t) = b_2 t + b_3,\\
&\Omega^1(X,t,x) = a_2 x + \beta_3 t + \beta_4(X),\\
&\Omega^2(\theta) = \mu_2\theta,
\end{aligned} \tag{2}$$

where $a = \frac{1}{2}(b_2 - a_2 - \frac{c}{2})$ and $a_2, b_2, b_3, \beta_3, \mu_2, c, c_1$ are arbitrary constants and β_4 is an arbitrary function of X.
The free energy function F should have the form

$$F(X,p,\theta) = f(X,p)\theta^2 + \phi(X,p), \tag{3}$$

where the functions f and ϕ should fulfil the partial differential equations

$$(5a + 2\mu)f + (aX + c_1)f_X + (\beta'(X) + pb)f_p = 0, \tag{4}$$

$$5a\phi + (aX + c_1)\phi_X + (\beta'(X) + pb)\phi_p = 0, \tag{5}$$

where

$$b = -\frac{1}{2}(b_2 - 3a_2 - \frac{c}{2}), \quad \mu = \mu_2, \quad \beta(X) = \beta_4(x). \tag{6}$$

In the present paper we focus our attention on non-linear homogeneous thermoelasticity the field equations of which can be written in the form of balance equations as

$$\frac{\partial}{\partial X}(\frac{\partial F}{\partial p}) - \frac{\partial(\rho_0 v)}{\partial t} = 0 \tag{7}$$

$$\frac{\partial}{\partial X}(k(X)\frac{\partial \theta}{\partial X})+\frac{\partial}{\partial t}(\theta\frac{\partial F}{\partial \theta})-\frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial t} = 0, \tag{8}$$

where ρ_0 is the mass density which is now considered constant throughout the body. The free energy function does not depend any more on space variable X, namely, $F = F(p, \theta)$. It is important to note that although we assume homogeneity in material properties, we do not consider constant thermal conductivity as one could expect. on the contrary, we continue to consider non-homogeneity with respect to thermal properties because as it will be apparent later accepting constant conductivity will cause cancellation of every non-trivial symmetry. Also, it is worthwhile to remark that actually we do not have a single system of partial differential equations but a class of systems depending on the particular form of free energy function F.

The main step towards the symmetries of the system is to obtain the isovector field, that is a vector field of the form

$$V = \omega^1\frac{\partial}{\partial X} + \omega^2\frac{\partial}{\partial t} + \Omega^1\frac{\partial}{\partial x} + \Omega^2\frac{\partial}{\partial \theta} \tag{9}$$

To find out the isovector field (infinitesimal generator) of the system we follow a method proposed by Suhubi [7]. This method was used by Suhubi himeself to study the similarity solutions for plane waves in hyperelastic materials [8]. Also, the author of this article used the method to study the non-homogeneous, one-dimensional problem of thermoelasticity. The interested reader can find in [1] the formulation of the problem and all technical details which are not repeated here. For the time being, the only difference with [1] is that f as well as ρ_0 do not depend on the space variable X, that means we adopt what is obtained in [1] for the isovector field, namely eqs (3.33):

$$\omega^1 = \omega^1(X), \quad \omega^2 = \omega^2(t), \quad \Omega^1 = \Omega^1(X, t, x), \quad \Omega^2 = \Omega^2(X, t, \theta).$$

Furthermore, we have to adopt equations (3.34), (3.35), (3.36) and (3.39) of [1] provided that all partial derivatives with respect to X are taken zero. An elaboration of these equations in a manner analogous to [1], will give us the following equations:

$$(-\omega^{1'}+2\dot{\omega}^2-\frac{\partial\Omega^1}{\partial x}+\frac{\partial\Omega^2}{\partial\theta})F_{p\theta}+\Omega^2F_{p\theta\theta}+(\frac{\partial\Omega^1}{\partial X}-p\omega^{1'}+p\frac{\partial\Omega^1}{\partial x})F_{pp\theta}=0, \quad (10)$$

$$\frac{\partial\Omega^2}{\partial X}F_{p\theta}-\rho_0\frac{\partial^2\Omega^1}{\partial t^2}+(\frac{\partial^2\Omega^1}{\partial X^2}-p\omega^{1''}+2p\frac{\partial^2\Omega^1}{\partial X\partial x})F_{pp}=0, \quad (11)$$

$$[(k'/k)\omega^1+\dot{\omega}^2](\theta F_{\theta\theta}+F_\theta)=-\frac{\partial\Omega^2}{\partial\theta}F_\theta-\omega^{1'}F_\theta+\Omega^2(F_{\theta\theta}+\theta F_{\theta\theta\theta})$$
$$+\theta F_{\theta\theta p}(\frac{\partial\Omega^1}{\partial X}-p\omega^{1'}+p\frac{\partial\Omega^1}{\partial x}), \quad (12)$$

$$k\frac{\partial^2\Omega^2}{\partial X^2}+k'\frac{\partial\Omega^2}{\partial X}+\frac{\partial\Omega^2}{\partial t}\theta F_{\theta\theta}+(\frac{\partial^2\Omega^1}{\partial X\partial t}+p\frac{\partial^2\Omega^1}{\partial x\partial t})\theta F_{\theta p}=0, \quad (13)$$

$$[(2\omega^{1'}-2\dot{\omega}^2]F_{pp}-\Omega^2F_{pp\theta}-(\frac{\partial\Omega^1}{\partial X}-p\omega^{1'}+p\frac{\partial\Omega^1}{\partial x})F_{ppp}=0, \quad (14)$$

$$\Big[(k'/k)\omega^1-\omega^{1'}+\dot{\omega}^2+\frac{\partial\Omega^2}{\partial\theta}-\frac{\partial\Omega^1}{\partial x}\Big]\theta F_{\theta p}=\Omega^2(F_{\theta p}+\theta F_{\theta\theta p})+$$
$$(\frac{\partial\Omega^1}{\partial X}-p\omega^{1'}+p\frac{\partial\Omega^1}{\partial x})\theta F_{\theta pp}. \quad (15)$$

Moreover, we obtain straightforwardly from eq. (3.34) and (3.35) of [1], some information concerning the components of the isovector field, namely the relations:

$$\frac{\partial^2\Omega^1}{\partial x^2}=0, \quad \frac{\partial^2\Omega^1}{\partial x\partial t}=\frac{1}{2}\ddot{\omega}^2, \quad \frac{\partial^2\Omega^1}{\partial X\partial x}=0,$$
$$\frac{\partial^2\Omega^2}{\partial\theta^2}=0, \quad \frac{\partial^2\Omega^2}{\partial X\partial\theta}=\frac{1}{2}(-[(k'/k)\omega^1]'+\omega^{1''}). \quad (16)$$

Next, a detailed elaboration of these equations (see Appendix) will give the following information concerning the general form of isovector field, i.e., the symmetries of eqs. (7)–(8) as well as some constraints on the general form of F and k. Hence the isovector field will take the form:

$$\omega^1(X)=c_1X+c_2,$$
$$\omega^2(t)=b_1t^2+b_2t+b_3,$$
$$\Omega^1(X,t,x)=b_1tx+a_2x+\beta_1tX+\beta_2t+\beta_3X+\beta_4,$$
$$\Omega^2(t,\theta)=-4b_1t\theta+\mu_1\theta, \quad (17)$$

where $c_1, c_2, b_1, b_2, b_3, a_2, \beta_1, \beta_2, \beta_3, \beta_4, \mu_1$ are arbitrary conctants, thus for the time being, we have obtained a 11-parameter group of infinitesomal transformations. It is important to warn the reader that the parameters in (17) are not related necessarily with the corresponding ones of (2). Furthermore, it is noted that the number of the parameters will reduce as it will be apparent in next section.
The free energy function is constrained to have the form

$$F(p, \theta) = f(p)\theta^2 + \phi(p), \tag{18}$$

where f and ϕ are arbitrary functions and the conductivity function should fulfil the equations

$$(k'/k)\omega^1 = c, \tag{19}$$

where c is an arbitrary constant and

$$(k'/k)\omega^1 + 3\dot{\omega}^2 - 2\omega^{1'} + \frac{\partial \Omega^2}{\partial \theta} - 2\frac{\partial \Omega^1}{\partial X} = 0. \tag{20}$$

3 The Isovector Field

In this section, after an exchausting elaboration of eqs (11)—(20), we will give the main result, that is the isovector field for homogeneous thermoelasticity. The way we follow to elaborate them demands that equations (11)—(15) admit isovector field (17). This will result in new equations to which functions f and ϕ should obey, which in turn will provide new relationships for the parameters and the isovector itself. Before we apply this procedure, we insert (17) into relationship (20) to obtain

$$c = -3b_2 + 2c_1 - \mu_1 + 2a_2. \tag{21}$$

Next, inserting isovector field (17) into eq. (11) we get

$$\begin{gathered}(-c_1 + 2(2b_1 t + b_2) - (b_1 t + a_2) + (-4b_1 t + \mu_1)2\theta f_p + \\ (-4b_1 t + \mu_1)2\theta f_p + (\beta_1 t + \beta_3 - pc_1 + p(b_1 t + a_2)2\theta f_p = 0,\end{gathered}$$

from which we obtain

$$(-c_1 + 2b_2 - a_2 + 2\mu_1)f_p + (\beta_3 - p(c_1 - a_2))f_{pp} = 0, \tag{22}$$

$$-5b_1 f_p + (\beta_1 + pb_1)f_{pp} = 0. \tag{23}$$

Doing the same with eq. (12) will give nothing because, provided eqs. (17), it (eq. (12)) is fulfilled identically.

Equation (13) with the help of relationship (21) will give

$$(4b_2 - 5c_1 + 2\mu_1 - 4a_2)f + (\beta_3 - p(c_1 - a_2))f_p = 0, \tag{24}$$

$$-4b_1 f + (\beta_1 + pb_1)f_p = 0. \tag{25}$$

The same procedure applied on eq. (14) will give eq. (25) too.

Following the same line of argument, eq. (15) in turn will become

$$\begin{aligned}\theta t^2[-4b_1 f_{pp} + (\beta_1 + pb_1)f_{ppp}] + t[4b_1\phi_{pp} + (\beta_1 + pb_1)f_{ppp}] +\\ \theta^2[(-2c_1 + 2b_2 + 2\mu_1)f_{pp} + [\beta_3 - p(c_1 - a_2)]f_{ppp}] +\\ [(-2c_1 + 2b_2)\phi_{pp} + [\beta_3 - p(c_1 - a_2)]\phi_{ppp}] = 0,\end{aligned}$$

from which we obtain

$$(-2c_1 + 2b_2 + 2\mu_1)f_{pp} + [\beta_3 - p(c_1 - a_2)]f_{ppp} = 0, \tag{26}$$

$$-4b_1 f_{pp} + (\beta_1 + pb_1)f_{ppp} = 0, \tag{27}$$

$$4b_1\phi_{pp} + (\beta_1 + pb_1)f_{ppp} = 0, \tag{28}$$

$$(-2c_1 + 2b_2)\phi_{pp} + [\beta_3 - p(c_1 - a_2)]\phi_{ppp} = 0. \tag{29}$$

The last one of the equations we treat, namely eq. (16), does not have any interest because it provides eqs. (22)—(23) which we have already taken from eq. (11). To clear up the situation we remark that all the equations (22)—(27) concern function f, so they should be valid simoultaneously. Demanding this we can obtain new relationships between the parameters, which will lead to the reduction of parameters number that is the modification of the isovector field itself.

It is easy for one to see that equations (22), (23) and (26), (27) respectively are compatible, by means that the former provides the latter by a simple derivation. Hence, we treat only eqs. (22), (23) which in turn should be compatible to eqs. (24)—(25). Differentiating eq. (25) we take

$$-3b_1 f_p + (\beta_1 + pb_1)f_p = 0. \tag{30}$$

Comparing this with eq. (23), we obtain

$$b_1 = 0, \tag{31}$$

and

$$\beta_1 = 0, \quad \text{or} \quad f \quad \text{constant}. \tag{32}$$

Obviously, choosing f to be constant, we conclude a free energy function of the form

$$F(p,\theta) = \lambda\theta^2 + \phi(p), \quad \lambda \quad \text{constant}$$

which does not provide coupled thermoelasticity, hence we proceed adopting the first choice, i.e.,

$$\beta_1 = 0 \tag{33}$$

Differentiating now eq. (24), we obtain

$$(4b_2 - 6c_1 + 2\mu_1 - 3a_2)f_p + (\beta_3 - p(c_1 - a_2))f_{pp} = 0,$$

which after comparison with eq. (22) will give

$$c_1 = \frac{2}{5}(b_2 - a_2). \tag{34}$$

After obtaining the relationships (31), (33) and (34) betwwen the parameters, it remains a unique differential equation that f should fulfil

$$(4b_2 + \mu_1 - a_2)f + (\beta_3 - \frac{p}{5}(2b_2 - 7a_2))f_p = 0. \tag{35}$$

The same is true for function ϕ. Inserting eqs. (30), (32) and (33) into eqs. (28) and (29), the former is fulfilled identically and the latter takes the form

$$\frac{2}{5}(3b_2 + 2a_2)\phi_{pp} + (\beta_3 - \frac{p}{5}(2b_2 - 7a_2))\phi_{ppp} = 0. \tag{36}$$

It is now apparent that the only meaningful choice regarding proposition (32) is $\beta_1 = 0$. Otherwise, we will necessarily conclude that $\phi =$ constant which does not make any sense for thermoelasticity. Actually, there will rise a free energy function depending only on temparature field θ, thus appropriate for a linear heat conduction theory for rigid media.

Let us return to the differential equation (19) governing the behavior of the conductivity function k. We recall this equation as it is

$$(k'/k)\omega^1 = c, \tag{37}$$

noting that c is not any more an arbitrary constant, but it is linked with the parameters of the symmetry group through the relationship

$$c = -\frac{11}{5}b_2 + \frac{6}{5}a_2 - \mu_1. \tag{38}$$

Equation (38) directly rises from eqs. (21) and (34).

We are passing now to the isovector field in which we enter all information about the parameters, inserting eqs. (31), (33) and (34) into eq. (17)

$$\begin{aligned}
&\omega^1(X) = \frac{2}{5}(b_2 - a_2)X + c_2, \\
&\omega^2(t) = b_2 t + b_3, \\
&\Omega^1(X, t, x) = a_2 x + \beta_2 t + \beta_3 X + \beta_4, \\
&\Omega^2(t, \theta) = \mu_1 \theta.
\end{aligned} \tag{39}$$

Thus, we finally obtain a 8-parameter group of transformations which we will examine in detail in the next section. For the time being, summarizing our main conclusion we can claim:

The symmetry group admitted by the system of one-dimensional, non-linear, homogeneous thermoelasticity (7), (8), is given by (39) provided that the free energy function is of the form

$$F(p, \theta) = f(p)\theta^2 + \phi(p), \tag{40}$$

where the functions f and ϕ fulfil the differential equations (35) and (36), respectively and the heat conductivity function k is governed by the differential equation (36) - 37).

4 Special Cases of Symmetries

The results presented in last section continue to be so general that they can not let us scrutinize particular cases of probably practical interest. This is due to the fact that our main equations (7) and (8) are not a sole system; actually they make up a class of equations, depended upon free energy function F. Hence, for every choice of F we take a seperate member of the class. To talk about symmetries one must first talk about the form of function F. This is already apparent due to the fact that our main result on admissible symmetries (39) depends on the class of functions F having the form (40). To proceed further one need to have particular F, or to put further constraints on the free energy function. This is exactly our next step.

4.1 The funcion f is arbitrary

Letting full arbitrariness to f means the differential equation (35) is valid for every f, hence the coefficients of the equation should be

$$\beta_3 = 0, \quad b_2 = \frac{7}{2}a_2, \quad \mu_1 = -\frac{5}{2}a_2. \tag{41}$$

After eqs. (41), the isovector field (39) becomes

$$\begin{aligned} \omega^1(X) &= a_2 X + c_2, \\ \omega^2(t) &= \frac{7}{2}a_2 t + b_3, \\ \Omega^1(t,x) &= a_2 x + \beta_2 t + \beta_4, \\ \Omega^2(\theta) &= -\frac{5}{2}a_2\theta \end{aligned} \tag{42}$$

and the differential equations for ϕ and k become

$$\frac{23}{5}a_2\phi_{pp} = 0 \Rightarrow \phi_{pp} = 0, \quad \text{for} \quad a_2 \neq 0$$

and

$$(k'/k)\omega^1 = -4a_2.$$

Hence we obtain

$$k(X) = (a_2X + c_2)^{-4}, \quad \phi(p) = \phi_1 p + \phi_2, \tag{43}$$

where c_2, ϕ_1 and ϕ_2 are arbitrary constants. Hence, for the case under study our initial system (7)—(8) is constrained to have the form

$$f''(p)\theta^2\frac{\partial^2 x}{\partial X^2} + 2f'(p)\theta\frac{\partial\theta}{\partial X} - \rho_0\frac{\partial^2 x}{\partial t^2} = 0, \tag{44}$$

$$k'(X)\frac{\partial\theta}{\partial X} + k(x)\frac{\partial^2 x}{\partial X^2} + 2f(p)\theta\frac{\partial\theta}{\partial t} + 2f'(p)\theta^2\frac{\partial^2 x}{\partial X\partial t} = 0. \tag{45}$$

We summarize what we have found for this particular case in the following statement:

If the differential equations (44)—(45) admit the symmetries given by isovector field (42), for arbitrary f then the function k will be necessarily of the form (43a).

Looking at the isovector field (42) we can recognize that the parameter a_2 gives the scaling (symmetry) and c_2, b_3 and β_4 are related with translations with respect to X, t and x, respectively. It is worthwhile to further examine the symmetry of scalings, thus to set $a_2 \neq 0$ and $c_2 = b_3 = \beta_2 = \beta_4 = 0$. In other words we examine the particular infinitesimal generator

$$V = X\frac{\partial}{\partial X} + \frac{7}{2}t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{5}{2}\theta\frac{\partial}{\partial \theta},$$

or the particular Lie group of finite transformations of scaloing type

$$X^* = e^{\epsilon}X, \quad t^* = e^{\frac{7}{2}\epsilon}t, \quad x^* = e^{\epsilon}x, \quad \theta^* = e^{\frac{7}{2}\epsilon}\theta. \tag{46}$$

The previous analysis secures that the transformation group (46) is admitted by PDE (44)—(45) provided the conductivity function k is of the particular form

$$k(X) = CX^{-4}. \tag{47}$$

4.2 The function ϕ is arbtrirary

We let now eq. (36) be valid for every ϕ which results

$$b_2 = -\frac{2}{3}a_2, \quad b_2 = \frac{7}{2}a_2, \quad \beta_3 = 0. \tag{48}$$

Relations (48) make sense only if $a_2 = b_2 = 0$. Hence the isovector field (39) becomes

$$\begin{aligned} &\omega^1(X) = c_2, \\ &\omega^2(t) = b_3, \\ &\Omega^1(t, x) = \beta_2 t + \beta_4, \\ &\Omega^2(\theta) = \mu_1\theta \end{aligned} \tag{49}$$

and eq. (35) takes the form

$$2\mu_1 f = 0.$$

If we want to keep the symmetry related to the parameter μ_1, we must necessarily consider $f = 0$, which in turn means that

$$F = F(p) = \phi(p), \quad \phi \quad \text{arbitrary},$$

which, certainly, does not lead to any kind of thermoelasticity. In order to have thermoelasticity, we must put $\mu_1 = 0$, hence f *is an arbitrary function* and the isovector field becomes

$$\begin{aligned}
&\omega^1 = c_2, \\
&\omega^2 = b_3, \\
&\Omega^1(t) = \beta_2 t + \beta_4, \\
&\Omega^2 = 0
\end{aligned} \tag{50}$$

and the free energy funcion will take the form

$$F(p, \theta) = f(p)\theta^2 + \phi(p), \tag{51}$$

where f and ϕ are arbitrary functions.

We examine now the specific symmetry corresponding to the parameter $\beta_2 \neq 0$, which arises within the case 4.1 as well. (It is worthwhile to examine whether the case 4.1 for $\beta_2 \neq 0$ will give us the arbitrariness of ϕ which we enjoy in the present case). In other words we discuss about the symmetry

$$X^* = X, \quad t^* = t, \quad x^* = x + t\epsilon, \quad \theta^* = \theta. \tag{52}$$

Recalling now that equation $k(X)$ should obey

$$(k^{'}/k)\omega^1 = -4a_2,$$

it is easy to conclude that $k' = 0$, thus the function $k(X)$ *is becoming a simple constant.* After that the field equations of thermoelasticity (i.e., eqs. (7)—(8) will take the form:

$$[f''(p)\theta^2 + \phi''(p)]\frac{\partial^2 x}{\partial X^2} + 2f'(p)\theta\frac{\partial \theta}{\partial X} - \rho_0\frac{\partial^2 x}{\partial t^2} = 0, \tag{53}$$

$$k\frac{\partial^2 x}{\partial X^2} + 2f(p)\theta\frac{\partial \theta}{\partial t} + 2f'(p)\theta^2\frac{\partial^2 x}{\partial X \partial t} = 0. \tag{54}$$

Concluding, we can claim that *the symmetry group given by eqs. (52) is the unique symmetry that is admitted by full homogeneous (k, ρ_0 constants) non-linear thermoelastic materials governed by eqs. (53)—(54).*

4.3 The function k is arbitrary

In order the function k to be an arbitrary one, i.e., every function k to satisfy the differential equation (37), we must put

$$\omega_1 \equiv 0, \quad c = 0, \tag{55}$$

from which we conclude straightforwordly

$$b_2 = a_2, \quad c_2 = 0. \tag{56}$$

Furthermore, in virtue of eq. (38) we obtain

$$\mu_1 = -a_2. \tag{57}$$

So, after eqs. (56)—(57) we obtain for the comonents of the isovector field:

$$\begin{aligned}
&\omega^1 = 0,\\
&\omega^2(t) = a_2 t + b_3,\\
&\Omega^1(X,t,x) = a_2 x + \beta_2 t + \beta_3 X + \beta_4,\\
&\Omega^2(\theta) = -a_2\theta.
\end{aligned} \tag{58}$$

Coming back now, to eqs. (35)—(36) which for the case under study they take the specific form

$$\begin{aligned}
2a_2 f - (\beta_3 + pa_2) f_p = 0,\\
2a_2 \phi_{pp} + (\beta_3 + pa_2)\phi_{ppp} = 0.
\end{aligned} \tag{59}$$

After that, the field equations (7)–(8) for the four–parameter symmetry group (58) take the form

$$[f''(p)\theta^2 + \phi''(p)]\frac{\partial^2 x}{\partial X^2} + 2f'(p)\theta\frac{\partial \theta}{\partial X} - \rho_0\frac{\partial^2 x}{\partial t^2} = 0, \tag{60}$$

$$k'(X)\frac{\partial \theta}{\partial X} + k(X)\frac{\partial^2 \theta}{\partial X^2} + 2f(p)\theta\frac{\partial \theta}{\partial t} + 2f'(p)\theta^2\frac{\partial^2 x}{\partial X \partial t} = 0. \tag{61}$$

The most interesting symmetry for the case under discussion seems to be the corresponding one to the parameter a_2:

$$V = \frac{\partial}{\partial X} + t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \theta\frac{\partial}{\partial \theta},$$

or in the form of a transformation group

$$X^* = X, \quad t^* = e^{\epsilon}t, \quad x^* = e^{\epsilon}x, \quad \theta^* = e^{-\epsilon}\theta. \tag{62}$$

For this particular symmetry, the differential equations (59)—(60) take the form

$$\begin{aligned}
2f - pf_p = 0,\\
2\phi_{pp} + p\phi_{ppp} = 0,
\end{aligned} \tag{63}$$

Thus the functions f and ϕ should have the form

$$f(p) = C_1 p^2, \quad \phi(p) = C_2 \ln p + C_3 p + C_4, \tag{64}$$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

5 Similarity Solutions

The next question is whether the symmetries we have found in last section, give any of the so-called group invariant solutions. We remind here that an invariant solution for a group of transformations admitted by the field equations, is nothing but a solution of the field equations which moreover is invariant under this group. The well-known similarity solutions are invariant solutions corresponding to the particular case of a scaling group. That means that a solution (x, θ) of the field equations (7)—(8) is an invariant one, for a given symmetry in the form of eq. (9), with $\omega^i \neq 0, i = 1, 2$, if they fulfil the differential equations

$$\begin{aligned} \omega^1 \frac{\partial x}{\partial X} + \omega^2 \frac{\partial x}{\partial t} &= \Omega^1, \\ \omega^1 \frac{\partial \theta}{\partial X} + \omega^2 \frac{\partial \theta}{\partial t} &= \Omega^2. \end{aligned} \tag{65}$$

So, we have to check all symmetries deriving in last section under this requirement

• *Symmetry given by eqs. (46).*

In this case the PDEs (65) take the form

$$X \frac{\partial x}{\partial X} + \frac{7}{2} t \frac{\partial x}{\partial t} = x, \tag{66}$$

$$X \frac{\partial \theta}{\partial X} + \frac{7}{2} t \frac{\partial \theta}{\partial t} = -\frac{5}{2} \theta. \tag{67}$$

Their solutions will be

$$x(X, \xi) = u(\xi) X, \quad \theta(x, \xi) = v(\xi) X^{-\frac{5}{2}}, \tag{68}$$

where $\xi = X t^{-\frac{2}{7}}$ is the similarity variable. Thus the function given by eqs. (68) will be the similarity solution of field equations (44)—(45). In order to check this, we have to carry out some calculations:

$$p = \frac{\partial x}{\partial X} = u'(\xi) \frac{\partial \xi}{\partial X} X + u(\xi) = u'(\xi) t^{-\frac{2}{7}} X + u(\xi) \Rightarrow$$

$$p = \xi u'(\xi) + u(\xi). \tag{69}$$

In the same manner we obtain

$$\frac{\partial^2 x}{\partial X^2} = t^{-\frac{2}{7}}(2u'(\xi) + \xi u''(\xi)), \tag{70}$$

$$\frac{\partial x}{\partial t} = -\frac{2}{7}u'(\xi)\xi^2 t^{-\frac{5}{7}}, \tag{71}$$

$$\frac{\partial^2 x}{\partial t^2} = \frac{4}{49}u''(\xi)\xi^3 t^{-\frac{12}{7}} + \frac{18}{49}u'(\xi)\xi^2 t^{-\frac{12}{7}}, \tag{72}$$

$$\frac{\partial^2 x}{\partial X \partial t} = -\frac{2}{7}u''(\xi)\xi^2 t^{-1} - \frac{4}{7}u'(\xi)\xi t^{-1}, \tag{73}$$

$$\frac{\partial \theta}{\partial X} = v'(\xi)\xi X^{-\frac{7}{2}} - \frac{5}{2}v(\xi)X^{-\frac{7}{2}}, \tag{74}$$

$$\frac{\partial^2 \theta}{\partial X^2} = [v''(\xi)\xi^2 - 5v'(\xi)\xi + \frac{35}{4}v(\xi)]X^{-\frac{9}{2}}, \tag{75}$$

$$\frac{\partial \theta}{\partial t} = -\frac{2}{7}v'(\xi)\xi t^{-1}. \tag{76}$$

Substituting now eqs. (47) and (67)—(76) into eqs. (44)—(45), we obtain

$$f''(P)v^2(2\xi u' + \xi^2 u'') + f'(p)(2\xi v v' - 5v^2) - \rho_0(\frac{4}{49}\xi^9 u'' + \frac{18}{49}\xi^8 u') = 0, \tag{77}$$

$$C(\xi^2 v'' - 9\xi v' + \frac{75}{4}v) - \frac{4}{7}f(p)\xi^{\frac{9}{2}}vv' - \frac{4}{7}f'(p)v^2(\xi^{\frac{11}{2}}u'' + 2\xi^{\frac{9}{2}}u') = 0. \tag{78}$$

It is worthwhile to note that the above system consists of highly non-linear but *ordinary* differential equations as we expected to. This is an indirect confirmation that all the previous analysis was carried out correctly.

- *Symmetry given by eqs. (62)*

In this case, eqs. (65) will take the form

$$t\frac{\partial x}{\partial t} = x, \quad t\frac{\partial \theta}{\partial t} = -\theta. \tag{79}$$

Hence, the similarity solutions corresponding to the symmetry (62) must be of the form

$$x(X,t) = u(X)t, \quad \theta(x,t) = v(X)t^{-1}, \tag{80}$$

where u and v are arbitrary functions of X. To be sure that eqs. (80) are indeed similarity solutions we must check whether they reduce the number of the indepedent variableds of the system (60)—(61). Actually, we will check whether or not, eqs. (80) will transform the aforementioned system to a system of ordinary differential equations. Indeed, inserting eqs. (64) into PDEs (60)—(61) we obtain

$$C_1\theta^2\frac{\partial^2 x}{\partial X^2}+2C_1\theta\frac{\partial x}{\partial X}\frac{\partial \theta}{\partial X}-C_2(\frac{\partial x}{\partial X})^{-2}\frac{\partial^2 x}{\partial X^2}-\rho_0\frac{\partial^2 x}{\partial t^2}=0, \tag{81}$$

$$k'(X)\frac{\partial \theta}{\partial X}+k(X)\frac{\partial^2 \theta}{\partial X^2}+C_1\theta\frac{\partial \theta}{\partial t}(\frac{\partial x}{\partial X})^2+2C_1\theta^2\frac{\partial x}{\partial X}\frac{\partial^2 x}{\partial X\partial t}=0. \tag{82}$$

After that, we carry out the following calculations

$$\frac{\partial x}{\partial X}=u'(X)t, \quad \frac{\partial^2 x}{\partial X^2}=u''(X)t, \quad \frac{\partial x}{\partial t}=u(X),$$
$$\frac{\partial^2 x}{\partial t^2}=0, \quad \frac{\partial^2 x}{\partial X\partial t}=u'(X) \tag{83}$$

and

$$\frac{\partial \theta}{\partial X}=v'(X)t^{-1}, \quad \frac{\partial^2 \theta}{\partial X^2}=v''(X)t^{-1}, \quad \frac{\partial \theta}{\partial t}=-v(X)t^{-2}. \tag{84}$$

Thus, in our last step we have just to insert eqs. (83)–(84) into eqs. (81)–(82) to obtain the following system of ordinary differential equations

$$C_1v^2u''+2C_1vv'u'-C_2u'u''=0, \tag{85}$$
$$k'(X)v'+k(x)v''-C_1v^2u'+2C_1v^2u'^2=0. \tag{86}$$

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A Appendix: The Derivation of Eqs. (17)—(20)

We give here the technical details for the derivation of eqs.(17)—(20). Summarizing the information about isovector field at hand, namely equations (16), we can write

$$\omega^2(t) = b_1 t^2 + b_2 t + b_3,$$
$$\Omega^1(X,t,x) = (b_1 t + a_2)x + \beta(X,t),$$
$$\Omega^2(X,t,\theta) = [\lambda(X) + \mu(t)]\theta + \gamma(X,t), \tag{87}$$

where b_1, b_2, b_3 and a_2 are arbitrary constants, β, γ, μ are arbitrary functions and

$$\lambda(X) = \frac{1}{2}\int k(X)dX$$

If we differentiate now eqs. (10) and (11) with respect to X and θ respectively and after that subtract them from each other we can obtain by virtue of eqs. (16), for $F_{p\theta} \neq 0$

$$\omega^{1''} = 0 \Rightarrow \omega^1 = c_1 X + c_2. \tag{88}$$

Hence, up to this point we have proved the form of the first two components of the isovector field (17).

Differentiating eq. (14) with respect to θ, multiplying by θ, and then differentiating (15) with respect to p and finally subtracting from each other we take

$$\Big[\big[(k'/k)\omega^1 + 3\dot{\omega}^2 - 2\omega^{1'} + 2\frac{\partial\Omega^2}{\partial\theta} - 2\frac{\partial\Omega^1}{\partial x}\big]\theta - \Omega^2\Big]\theta F_{\theta pp} = 0.$$

Thus for $F_{\theta pp} \neq 0$, we obtain

$$(k'/k)\omega^1 + 3\dot{\omega}^2 - 2\omega^{1'} + \frac{\partial\Omega^2}{\partial\theta} - 2\frac{\partial\Omega^1}{\partial x} = 0, \tag{89}$$

which is the required eq. (20).

In what follows we elaborate carefully eq. (89); first we differentiate with respect to t and easily obtain

$$\mu(t) = -4b_1 t + \mu_1, \tag{90}$$

where μ_1 is an arbitrary constant. Next differentiating with respect to X, we obtain

$$(k'/k)\omega^1 = c. \tag{91}$$

Therefore, eq. (19) has derived. Moreover, with the aid of eq. (16) it gives

$$\frac{\partial^2 \Omega^2}{\partial X \partial \theta} = 0. \tag{92}$$

Inserting now eq. (89) into eq. (12) it becomes comparable to eq. (10). After proper differentiation this comparison results

$$\frac{\partial^2 \Omega^2}{\partial t^2} = 0. \tag{93}$$

Coming back to eqs. (10) and (11), differentiating them twice with respect to t , we can obtain

$$\frac{\partial^3 \Omega^1}{\partial X \partial t^2} = 0, \quad \frac{\partial^4 \Omega^1}{\partial t^4} = 0. \tag{94}$$

In the same line of argument, with the aid of eq. (89) we can make eqs. (10) and (15) to be comparable with each other. Thus for $F_{\theta p} \neq 0$ we obtain

$$\frac{\partial \Omega^2}{\partial \theta} - \Omega^2 = 0 \Rightarrow \Omega^2(t, \theta) = (-4b_1 t + \mu_1)\theta \tag{95}$$

The last step was obtained by virtue of eq. (92). After the last relation, we come back to eqs. (10) and (11) once more, differentiate with respect to X and t respectively to obtain

$$\frac{\partial^2 \Omega^1}{\partial X^2} = 0, \quad \frac{\partial^2 \Omega^1}{\partial t^2} = 0. \tag{96}$$

With eqs. (94) and (96) at hand the function β takes the form

$$\beta(X, t) = \beta_1 t X + \beta_2 t + \beta_3 X + \beta_4, \tag{97}$$

where $\beta_1, \beta_2, \beta_3$ and β_4 are arbitrary constants. After eqs. (95) and (97) the form of the remaining components of the isovector field (17) has been proved too.

Last, we apply the same way of elaboration to eqs. (13) and (15); that is we differentiate with respect to θ the former and with respect to t the latter and substitute each other. the result of this manipulation is

$$\theta F_{\theta\theta} - F_\theta = 0 \Rightarrow F(p, \theta) = f(p)\theta^2 + \phi(p), \tag{98}$$

where f and ϕ are arbitrary functions. Hence, eq. (18) has derived, too.

An Advisory System for Statistical Analysis

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ABSTRACT

In this paper an advisory system for statistical analysis (ASSA) is presented for use by non-statisticians. The objective is to encourage office-workers, who are non-statisticians to make, as far as possible, a correct statistical analysis with an efficient and effective way, by exploiting some attributes of their data. ASSA is data -driven and navigates the user through QUESTIONS and ANSWERS to the selection of the statistical methodology (statistical technique or model) which is appropriate for the data under hand.

1. INTRODUCTION

It is well known that statistical methodologies are used by a lot of scientists in almost every scientific field. Therefore today the demand is increasing for systems that can implement these methodologies. More specifically the need is for systems that can give advise about the appropriate statistical methodology on formatted data.

This increasing demand has been caused, firstly by the rapid development of personal computer (pc) and workstation (ws) technology in the recent years and the continuously declining cost of them, and secondly by the large number of statistical packages available. The proliferation of pc and ws- based office systems have expanded the production of statistical analysis in a wide range of activities (i. e. civil works, social sciences, agriculture, education e.t.c.)

The rapid development of pc's has had as a consequence the implementation of more advanced statistical techniques with a lot of details in the statistical packages. This fact confuses the non-statisticians and prohibit them to select the right statistical methodology.

The ASSA system aims exactly at this point. It gives correct advice for the appropriate statistical methodology since its knowledge base is restricted by the built-in theoretical limits of existing statistical theory. This system is efficient and effective for the user.

The benefits of this system are decision quality, increasing production, time saving, cost reduction and general improved services.

In the next paragraph we discuss the steps necessary for the statistical analysis whereas in the third paragraph we give a description of the ASSA system.

2. STEPS FOR THE ANALYSIS

Roughly speaking if someone wants to statistically analyze his data he has to follow the following steps.

STEP 1. *Selection of the statistical methodology*
This is the fundamental step in the analysis. A wrong selection of statistical methodology may lead to wrong conclusions or it may not reveal the complete structure of the data.

STEP 2. *Selection of the right statistical software*
Having decided about the statistical methodology to be used we have to select a statistical package to implement it. The selection is crucial because not all packages give the same (maximum) information to the user. We will illustrate this point with the following example. Suppose that our statistical methodology, selected in the previous step, is that of regression analysis. It is well known that the various forms of residuals (e.g. raw, standardized, deleted. e.t.c.) play quite an important role in such an analysis. So a statistical package which computes only the raw residuals is not satisfactory.

STEP 3. *Main analysis*
At this step the non-statistician needs some rules to follow in order to complete the analysis. For example in regression analysis one needs to know what the assumptions are about the error term and how they can be checked.

STEP 4. *Interpretation of the results*
This means not only the final conclusions but also the intermediate ones. The last ones are sometimes quite crucial because they may lead to a different model.
For example in regression analysis if the user does not interpret correctly the various graphs of the residuals then he may end up with a wrong model and consequently with wrong conclusions.

The above are shown graphically in figure 1.

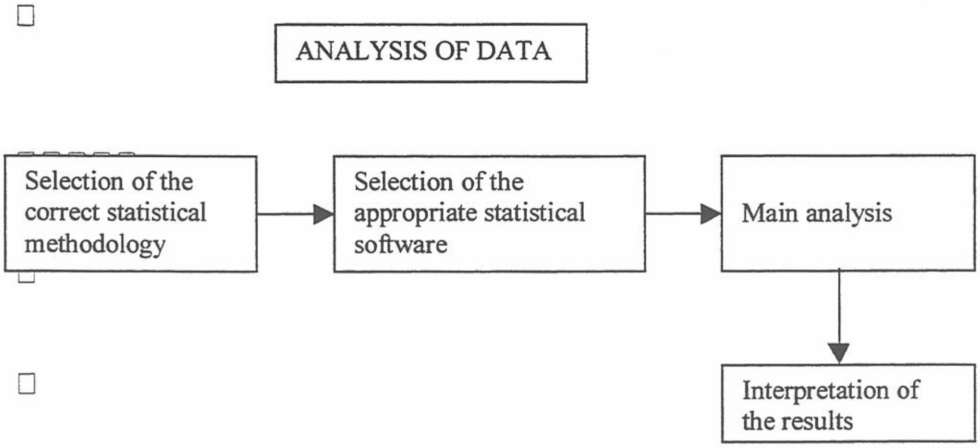


Figure 1. The steps necessary for statistically analyzing data

As we pointed out at the very beginning of this paper the user has a large number of statistical software to choose from. For a comparison between some of them see (να μπει αναφορά)

In the last few years some of the available statistical packages developed procedures along the lines of step1. (e.g SAS, S.P.S.S.). However their work in this area is not satisfactory because they mainly interpret the results and/or give some kind of a glossary and they do not suggest specific statistical methodology. To the authors knowledge there is no published work in this direction except for the work by Bill Trochim, which is available through the Internet in the address (http://trochim.human.cornell.edu/). Our system focuses on the first step although some kind of progress has been done on the third step too.

An ideal function for an expert statistical system of the kind described in this paper would be for the user to provide the necessary information in his natural language. The system, after doing a semantic analysis, recognize the user's needs and acts on the formatted data, then it response with all relevant information independently of the statistical package used by the user.

Up to now such a system does not exist. We hope that we shall be able to provide such a system in the near future.

DISCRIPTION OF THE ASSA

The ASSA, like most diagnostic systems, is data-driven. That means that the user provides some information- in the form of replies- to questions posed by the ASSA and then he receives the appropriate advice. All these questions and answers are kept on a logged window. The content in that window can be saved in a file for later use. For some questions, the user responds by selecting from a menu. In other cases it is necessary to type in the answer. The user has the option to ask the system to explain the question so that this becomes more clear. In addition the ASSA has the possibility to explain various statistical items, which exists in the question posed, in an html form. This can be done either by giving an appropriate definition or by giving an example.

A kind of such questions and possible answers are given in the following example.

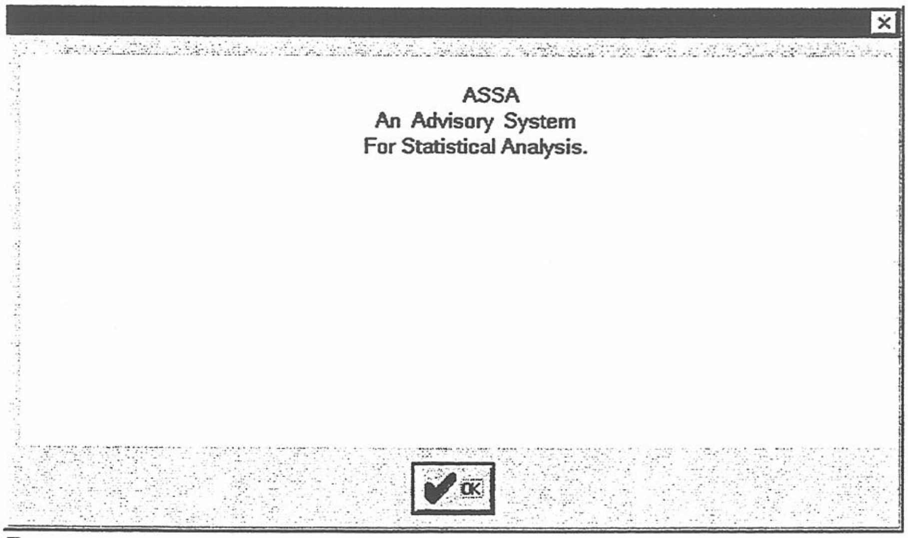
ASSA : *How many variables are involved in your analysis?*
User : Two (2)
ASSA : *Could any one of them be considered as a depended variable?*
User : Explain
ASSA : Your interest, at the moment, is focused on two variables of your data. If you are interested in studying e.g. the effect of the first variable on the second one, then the second variable is considered as dependent variable. As an example consider two variables e.g. WEIGHT and HEIGHT. If we want to see the effect of HEIGHT on WEIHGT then WEIGHT is the dependent variable. For more information you can see about that item on the HELP button.
User : Yes
ASSA : *Is that variable quantitative?*
User : Yes
ASSA : *Is the other variable (call it independent) quantitative too?*
User : Yes
ADVISE : *The suggested statistical methodology is that of the simple regression model.*

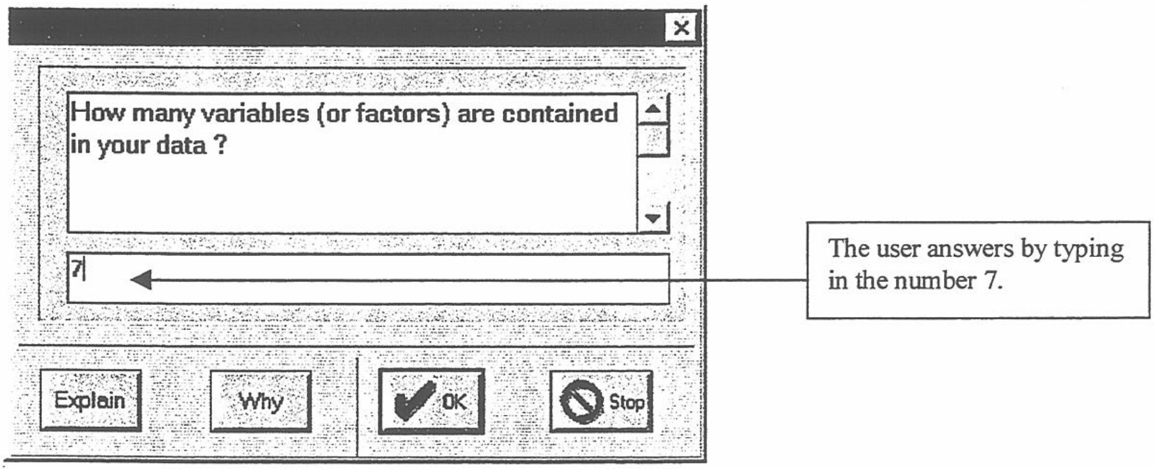
The other possibility was the goal-driven approach, which starts with a specific statistical methodology and tries to validate it. This it is not preferable since there exists great possibility for the user to pass through a large number of statistical methodologies and this may discourage him. Beside that it is also time consuming.

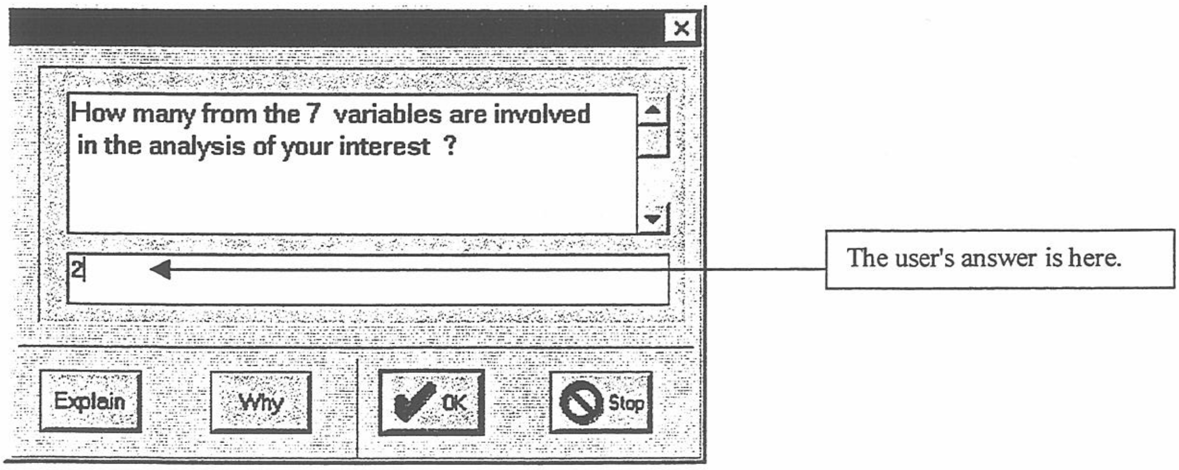
The knowledge base of ASSA contains about thirty rules. For example for the simple regression we have a rule like this:

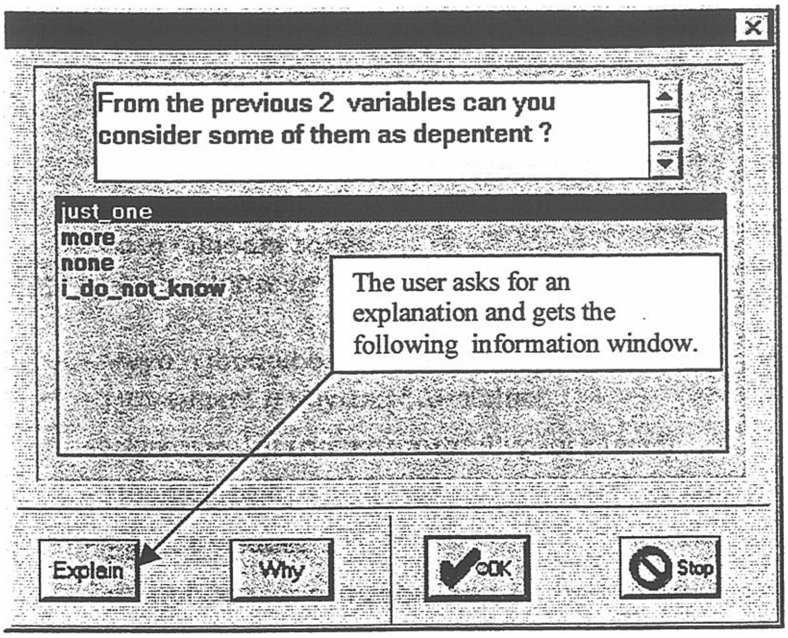
IF there are two (2) quantitative variables in the analysis
  AND one of them can be thought of as a dependent variable
THEN the model is that of the simple regression.

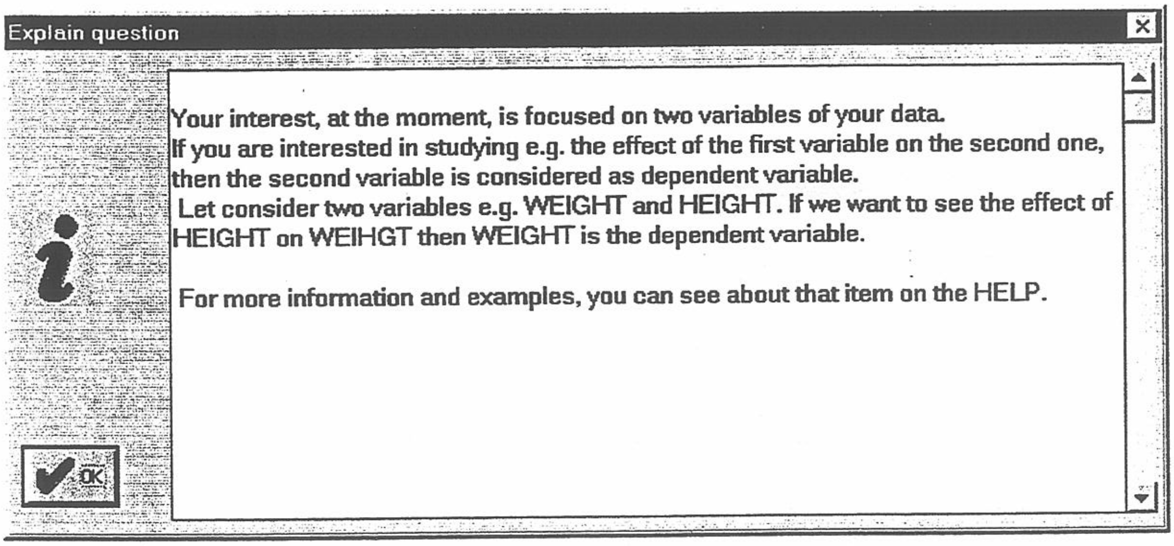
The ASSA system has been prototyped and tested with the shell ESTA
(Expert System shell for Text Animation).
An example using ASSA follows.


ASSA
An Advisory System
For Statistical Analysis.
OK

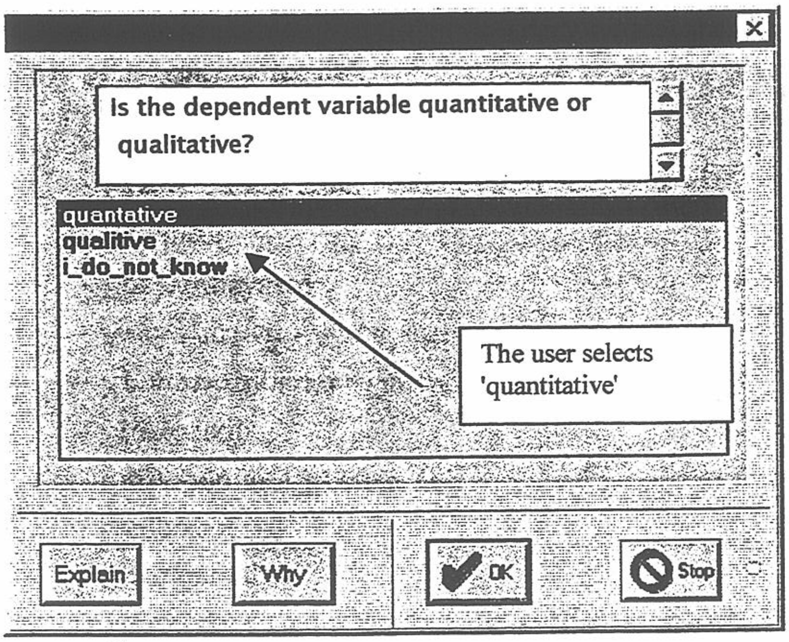

How many variables (or factors) are contained in your data ?
7
The user answers by typing in the number 7.
Explain
Why
OK
Stop

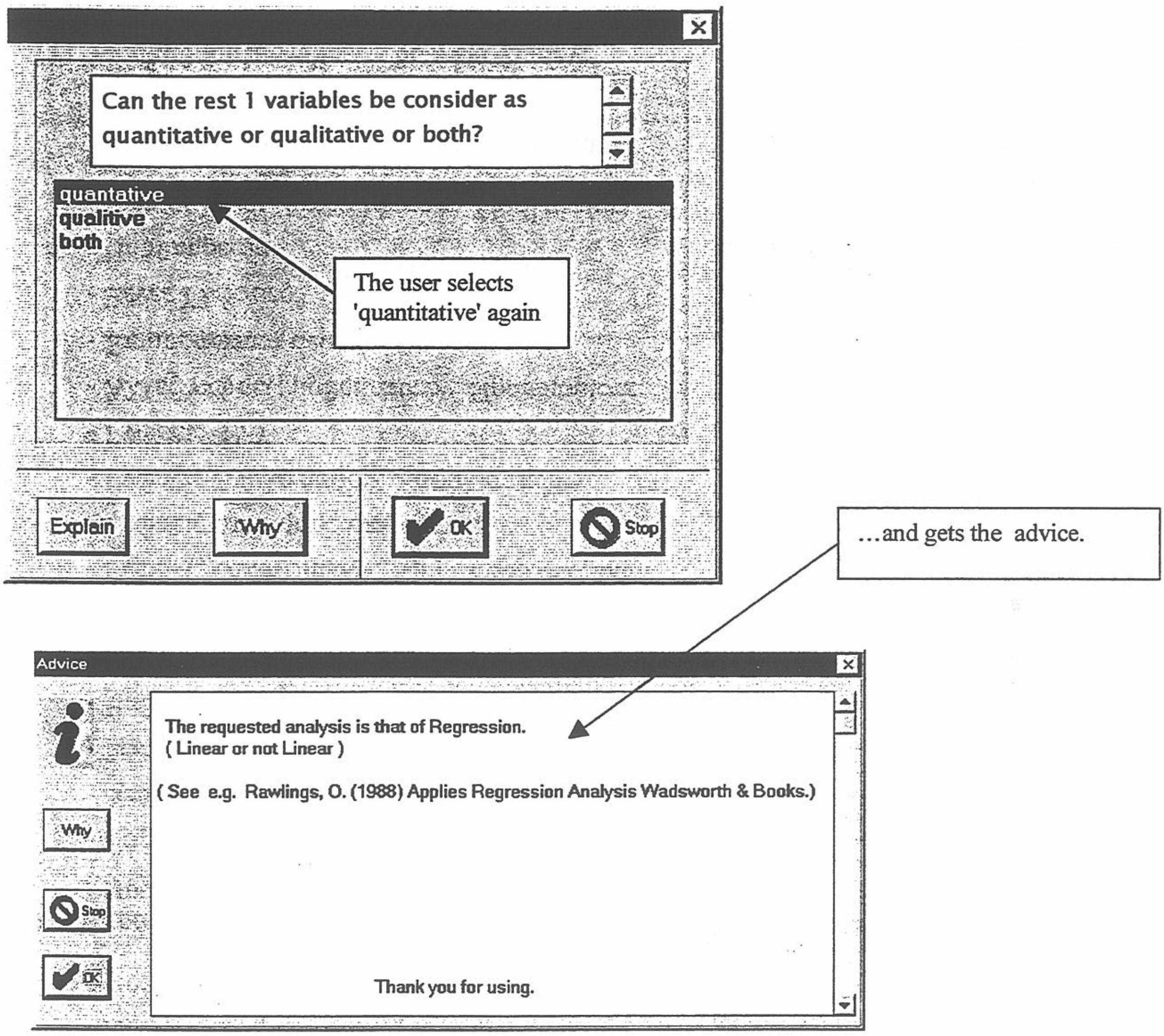

How many from the 7 variables are involved in the analysis of your interest ?
2
The user's answer is here.
Explain
Why
OK
Stop





By pressing the OK button in that window he gets the previous one where he selects the answer *'just_one'* and goes the following window.





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(http://trochim.human.cornell.edu/selstat/ssstart.htm)

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On a class of generalized (κ, μ)-contact metric manifolds

By

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Abstract. We classify the 3-dimensional generalized (κ, μ)-contact metric manifolds, which satisfy the condition $\|grad\kappa\|$ =const.$(\neq 0)$. This class of manifolds is determined by two arbitrary functions of one variable.

1. Introduction. The tangent sphere bundle, of a Riemaniann manifold of constant sectional curvature admits a contact metric structure (η, ξ, ϕ, g) such that the characteristic vector field ξ belongs to the (κ, μ)-nullity distribution, for some real numbers κ and μ. This means that the curvature tensor R satisfies the condition

$$(*) \qquad R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]$$

for any vector fields X and Y, where h denotes, up to a scaling factor, the Lie derivative of the structure tensor field ϕ in the direction of ξ. The class of contact metric manifolds which satisfy $(*)$ has been classified in all dimensions, see [2],[3],[4].

On the other hand, the existence of 3-dimensional contact metric manifolds M satisfying $(*)$, with κ, μ non constant smooth functions on M, has been proved in [7], through the construction of examples. (In [7] it is also proved that for dimensions greater than 3 such manifolds do no exist). This class of Riemannian manifolds seems to be particulary large and we call such a manifold a generalized (κ, μ)-contact metric manifold (generalized (κ, μ)-c.m.m., in short).

In §3 of the present paper we give more examples of generalized (κ, μ)-c.m.m., with the additional property $\|grad\kappa\|$ =constant. Moreover, we remark that the condition $\|grad\kappa\|$ =constant, remains invariant under a D-homothetic deformation. Hence for any positive real number we can construct at least two such manifolds. The existence of these examples has been our motivation for their study.

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Initially, we prove that there exist two types of generalized (κ,μ)-c.m.m. with $\|grad\kappa\|$ =constant $\neq 0$. Type A, where $\mu = 2(1-\sqrt{1-\kappa})$ and type B, where $\mu = 2(1+\sqrt{1-\kappa})$. Next, in §4 we prove that such a manifold is covered by a global chart, in the coordinates of which we determine the functions κ and μ. In §5 we globally construct these manifolds. Finally, introducing a second transformation in §6, we succeed each member of this class is obtained by the first two examples given in §3, under such a transformation and a D-homothetic deformation.

All manifolds are assumed to be connected.

2. Preliminaries. In this section we collect some basic facts about contact metric manifolds. We refer to [1] for more detailed treatment. A differential $(2m+1)$ -dimensional manifold M is called a contact metric manifold if it carries a global differential 1-form η such that $\eta \wedge (d\eta)^m \neq 0$ everywhere. It is known that a contact manifold admits an almost contact metric structure (η,ξ,ϕ,g), i.e. a global vector field ξ, which will be called the characteristic vector field, a (1,1)-tensor field ϕ and a Riemannian metric g such that $\eta(\xi) = 1, \quad \phi^2 = -Id + \eta \otimes \xi$ and $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$ for all vector fields X, Y on M. Moreover (η,ξ,ϕ,g) can be chosen such that $d\eta(X,Y) = g(X,\phi Y)$. The manifold M together with the structure tensors (η,ξ,ϕ,g) is called a contact metric manifold and it is denoted by $M(\eta,\xi,\phi,g)$. Following [1], we define on M the (1,1)-tensor fields h and l by $h = \frac{1}{2}(\mathcal{L}_\xi\phi)$ and $l = R(.,\xi)\xi$, where $\mathcal{L}_\xi$ is the Lie differentiation in the direction of ξ and R the curvature tensor. The tensor fields h, l are self adjoint and satisfy $h\xi = l\xi = 0, Trh = 0, Tr\phi h = 0, h\phi + \phi h = 0$,

$$Trl = g(Q\xi,\xi), \tag{1}$$

where Q is the Ricci operator. Since h anti-commutes with ϕ, if X is an eigenvector of h corresponding to the eigenvalue λ, then ϕX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$. If ∇ is the Riemannian connection of g, then $\nabla_\xi\phi = 0$,

$$\nabla_X\xi = -\phi X - \phi h X \quad (and\ so\ \nabla_\xi\xi = 0), \tag{2}$$

$$\nabla_\xi h = \phi - \phi l - \phi h^2. \tag{3}$$

Particularly, for the 3-dimensional case, the following formulas are valid ([6])

$$h^2 = (\frac{Trl}{2} - 1)\phi^2, \quad \frac{Trl}{2} \leq 1, \tag{4}$$

$$\sum_i (\nabla_{X_i} h)X_i = \phi Q\xi, \tag{5}$$

where $X_i, i = 1,2,3$, is an arbitrary orthonormal frame.

By a generalized (κ,μ)-contact metric manifold we mean a 3-dimensional contact metric manifold such that

$$R(X,Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \quad X, Y \epsilon \mathcal{X}(\mathcal{M}), \tag{6}$$

where κ, μ are smooth functions on M, independent of the choice of vector fields X and Y.

The formulas in the next Lemma are known (see [5], [7]). For the sake of completeness we will give the outline of their proofs.

Lemma 1. On any generalized (κ,μ)-c.m.m. the following formulas are valid

$$h^2 = (\kappa - 1)\phi^2, \quad \kappa = \frac{Trl}{2} \leq 1 \tag{7}$$

$$\xi\kappa = 0, \quad hgrad\mu = grad\kappa. \tag{8}$$

Moreover, if $\kappa \neq 1$ everywhere on M, then

(9) $\nabla_X \xi = -(\lambda+1)\phi X, \quad \nabla_{\phi X}\xi = (1-\lambda)X,$

(10) $\nabla_\xi X = -\frac{\mu}{2}\phi X, \quad \nabla_\xi \phi X = \frac{\mu}{2}X, \quad \nabla_X X = \frac{\phi X\lambda}{2\lambda}\phi X, \quad \nabla_{\phi X}\phi X = \frac{X\lambda}{2\lambda}X,$

(11) $\nabla_{\phi X}X = -\frac{X\lambda}{2\lambda}\phi X + (\lambda-1)\xi, \quad \nabla_X \phi X = -\frac{\phi X\lambda}{2\lambda}X + (\lambda+1)\xi,$

(12) $[\xi, X] = (1+\lambda-\frac{\mu}{2})\phi X, \quad [\xi, \phi X] = (\lambda - 1 + \frac{\mu}{2})X,$

(13) $[X, \phi X] = -\frac{\phi X\lambda}{2\lambda}X + \frac{X\lambda}{2\lambda}\phi X + 2\xi,$

where $(\xi, X, \phi X)$ is a local orthonormal basis of eigenvectors of h, such that $hX = \lambda X, \quad \lambda = \sqrt{1-\kappa} > 0$.

Proof. Using (6), we easily get $R(\xi, X)Y = \kappa[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX]$ and so by the definition of Q and (1) we get $Q\xi = 2\kappa\xi$ and $Trl = 2\kappa$. This and (4) imply (7). Using (6), $Q\xi = 2\kappa\xi$ and the well known formula

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X$$
$$-g(QX,Z)Y - \frac{S}{2}(g(Y,Z)X - g(X,Z)Y)$$

for $Y = Z = \xi$, we get

$$Q = aI + b\eta \otimes \xi + \mu h, \tag{14}$$

where S is the scalar curvature, $a = \frac{1}{2}(S - 2\kappa)$ and $b = \frac{1}{2}(6\kappa - S)$. Using (6), $\phi^2 = -Id + \eta \otimes \xi$ and the definition of l we find $l = -\kappa\phi^2 + \mu h$. This, together with (3) , (7) and $h\phi + \phi h = 0$ give $\nabla_\xi h = \mu h\phi$. Differentiating $h^2 = (\kappa-1)\phi^2$ with respect to ξ and using $\nabla_\xi \phi = 0$ and the last equation we get the first equation of (8). Differentiating (14) with respect to an orthonormal basis $X_i, i = 1, 2, 3$, and using (2), $Trh\phi = 0$, $\phi\xi = h\xi = 0$, $Q\xi = 2\kappa\xi$ and (5) we find

$$\sum_i (\nabla_{X_i} Q)X_i = grada + (\xi b)\xi + hgrad\mu.$$

Comparing this with the well known formula $\sum_i (\nabla_{X_i} Q)X_i = \frac{1}{2} gradS$, we get $hgrad\mu = grad\kappa$. Relations (9) are immediate consequences of (2). The first two relations of (10) are obtained from (6), for $Y = \xi$, and the definition of the curvature tensor. Using (5) we get the last two relations of (10). Relations (11) follow from (9) and (10), while (12) and (13) are immediate consequences of (9)-(11). We denote that the existence of the local basis $(\xi, X, \phi X)$ is proved in [7].

3. Examples. 1. (Type A). We consider the 3-dimensional manifold $M = \{(x,y,z)\epsilon R^3 / z < 1\}$, where (x,y,z) are the standard coordinates in R^3. The 1-form $\eta = dx + 2ydz$ defines a contact structure on M with characteristic vector field $\xi = \frac{\partial}{\partial x}$. Let g, ϕ be the Riemannian metric and the (1,1)-tensor field given by

$$g = \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ -a & -b & 1+a^2+b^2 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & -a & ab \\ 0 & -b & 1+b^2 \\ 0 & -1 & b \end{pmatrix}$$

with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, where $a = -2y$ and $b = 2x\sqrt{1-z} + \frac{y}{4(1-z)}$. The tensor fields (η, ξ, ϕ, g) define a generalized (κ, μ)-contact metric manifold with $\kappa = z$ (and so $\|grad\kappa\| = 1$) and $\mu = 2(1 - \sqrt{1-z})$.

2. (Type B). On the manifold M of the previous example we define the tensor fields (η, ξ, ϕ, g) by $\eta = dx - 2ydz$, $\xi = \frac{\partial}{\partial x}$,

$$g = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & -b \\ a & -b & 1+a^2+b^2 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & -a & ab \\ 0 & b & -1-b^2 \\ 0 & 1 & -b \end{pmatrix}.$$

Then $M(\eta, \xi, \phi, g)$ is a generalized (κ, μ)-contact metric manifold with $\kappa = z$ and $\mu = 2(1 + \sqrt{1-z})$.

3. Let $M(\eta, \xi, \phi, g)$ be a contact metric manifold. By a D_a-homothetic deformation (see [8],[2]) we mean a change of structure tensors of the form

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta,$$

where a is a positive number.The curvature tensor R and the tensor h transform in the following manner [2]: $\bar{h} = \frac{1}{a}h$ and $a\bar{R}(X,Y)\bar{\xi} = R(X,Y)\xi + (a-1)^2[\eta(Y)X - \eta(X)Y] - (a-1)[(\nabla_X \phi)Y - (\nabla_Y \phi)X + \eta(X)(Y + hY) - \eta(Y)(X + hX)]$ for any X, Y. Additionally, it is well known [9, pp 446,447], that any 3-dimensional contact metric manifold satisfies $(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$. Using these we have that if $M(\eta, \xi, \phi, g)$ is a generalized (κ, μ)-c.m.m., then $M(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ is also a generalized $(\bar{\kappa}, \bar{\mu})$-c.m.m. with $\bar{\kappa} = \frac{\kappa + a^2 - 1}{a^2}$ and $\bar{\mu} = \frac{\mu + 2(a-1)}{a}$ ([7]). Therefore, if $M(\eta, \xi, \phi, g)$ satisfies $\|grad\kappa\|_g = d$ (const.), then $M(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ satisfies $\|grad\bar{\kappa}\|_{\bar{g}} = da^{-\frac{5}{2}}$. It follows from the fact that, if $(\xi, X, \phi X)$ is an orthonormal basis with respect to g, then $(\frac{1}{a}\xi, \frac{1}{\sqrt{a}}X, \frac{1}{\sqrt{a}}\phi X)$ is an orthonormal basis with respect to $\bar{g}$.

As a result of the above and examples 1,2, we have the following Proposition.

Proposition 2. For any positive number, there exist at least two generalized (κ,μ)-c.m.m. with $||grad\kappa||$ =constant $\neq 0$.

Remark 1. (i) Using the fact that, any generalized (κ,μ)-c.m.m. with $||grad\kappa||_g = d \neq 0$ (const.) is D_a-deformed in another generalized $(\bar{\kappa},\bar{\mu})$-c.m.m. with $||grad\bar{\kappa}||_{\bar{g}} = da^{-\frac{5}{2}}$, for any positive a and choosing $a = d^{\frac{2}{5}}$, it is enough to study those generalized (κ,μ)-c.m.m. with $||grad\kappa|| = 1$.
(ii) If $d = 0$, then κ is constant. Therefore, if $\kappa = 1$, then M is a Sasakian manifold [2], while for $\kappa \neq 1$, μ =constant [7].
(iii) A D_a-homothetic deformation preserves the type of a generalized (κ,μ)-contact metric manifold with $||grad\kappa||$ =const.

4. Main results. From now on, we suppose that $M(\eta,\xi,\phi,g)$ is a generalized (κ,μ)-contact metric manifold with $||grad\kappa|| = 1$. Because $hgrad\mu = grad\kappa$, we have $h \neq 0$ and so $\kappa \neq 1$ everywhere on M as it follows from (7). We denote by $(\xi, X, \phi X)$ a local orthonormal frame of eigenvectors of h such that $hX = \lambda X$, $\lambda = \sqrt{1-\kappa} > 0$. The next Lemma inform us that there exist 2-types of such manifolds.

Lemma 3. Let M be a generalized (κ,μ)-c.m.m. with $||grad\kappa|| = 1$. Then $\mu = 2(1-\lambda)$ or $\mu = 2(1+\lambda)$. In the first case (type A), the following are valid,
$X\kappa = 1, \phi X\kappa = 0, [\xi, X] = 2\lambda\phi X, [\xi,\phi X] = 0$ and $[X,\phi X] = -\frac{1}{4\lambda^2}\phi X + 2\xi$.
In the second case (type B), the following are valid,
$\phi X\kappa = 1, X\kappa = 0, [\xi, X] = 0, [\xi,\phi X] = 2\lambda X$ and $[X,\phi X] = \frac{1}{4\lambda^2} X + 2\xi$.

Proof. Using $\xi\kappa = 0$ and $||grad\kappa|| = 1$ we have

$$grad\kappa = (X\kappa)X + (\phi X\kappa)\phi X \tag{15}$$

and

$$(X\kappa)^2 + (\phi X\kappa)^2 = 1. \tag{16}$$

Differentiating (16) with respect to ξ and using (8) and (12) we get successively

$$(\xi X\kappa)(X\kappa) + (\xi\phi X\kappa)(\phi X\kappa) = 0$$
$$([\xi, X]\kappa)(X\kappa) + ([\xi,\phi X]\kappa)(\phi X\kappa) = 0$$
$$\lambda(X\kappa)(\phi X\kappa) = 0$$

and since $\lambda \neq 0$,

$$(X\kappa)(\phi X\kappa) = 0. \tag{17}$$

We consider the open sets

$$A = \{P \epsilon M/(X\kappa)(P) \neq 0\} \quad \text{and} \quad B = \{P \epsilon M/(\phi X\kappa)(P) \neq 0\}.$$

Because $||grad\kappa|| \neq 0$, we have $A \cap B = \emptyset$ and $A \cup B = M$. Moreover, by the connectness of M we get $A = M$ and $B = \emptyset$ or $B = M$ and $A = \emptyset$. We distinguish two cases.
Case 1. Let $A = M$. Then, (17) gives $\phi X\kappa = 0$. Using this, $X\kappa \neq 0$, $\xi\kappa = 0$, then the second of (12) gives $\mu = 2(1-\lambda)$ and $[\xi, \phi X] = 0$. Moreover, from (16) we have $X\kappa = \pm 1$. Without loss of generality we may assume that $X\kappa = 1$, differently we choose the basis $(\xi, -X, -\phi X)$. Differentiating $\lambda^2 = 1 - \kappa$ and using $X\kappa = 1$ and $\phi X\kappa = 0$ we get $X\lambda = -\frac{1}{2\lambda}$ and $\phi X\lambda = 0$. Substituting these in (12) and (13) we have $[\xi, X] = 2\lambda\phi X$ and $[X, \phi X] = -\frac{1}{4\lambda^2}\phi X + 2\xi$.
Case 2. Let $B = M$. Then $\phi X\kappa \neq 0$ and $X\kappa = 0$. Working as in case 1 we finally get $\mu = 2(1+\lambda)$, $\phi X\kappa = 1$, $[\xi, \phi X] = 2\lambda X$ and $[X, \phi X] = \frac{1}{4\lambda^2}X + 2\xi$. This completes the proof of the Lemma.

Remark 2. In the case of type A ($\mu = 2(1-\lambda)$), we have $X = grad\kappa$ and in the case of type B ($\mu = 2(1+\lambda)$), we have $X = -\phi grad\kappa$, as they follow from (15). Because the function κ is globally defined on M, we conclude that the orthonormal frame $(\xi, X, \phi X)$ of eingenvectors of h is globally defined on M.

Remark 2, leads us to the following Proposition.

Proposition 4. Any generalized (κ, μ)-c.m.m. with $||grad\kappa||$ =const.$\neq 0$ is parallelizable.

In the next Lemma, we construct a suitable chart, whose domain is the whole of the manifold.

Lemma 5. Let M be a generalized (κ, μ)-c.m.m. with $||grad\kappa|| = 1$. Then, there exists a chart (x, y, z) whose domain covers M. Moreover,$\kappa = z$, $z < 1$, everywhere on M.

Proof. According to Lemma 3 we distinguish two cases.
Case 1. Let $\mu = 2(1-\lambda)$. Because $[\xi, \phi X] = 0$, the distribution which is obtained by ϕX and ξ is integrable. So for any point $P \epsilon M$ there exists a chart $\{V, (\bar{x}, \bar{y}, \bar{z})\}$ at P, such that

$$\xi = \frac{\partial}{\partial \bar{x}}, \phi X = \frac{\partial}{\partial \bar{y}} \quad \text{and} \quad X = a\frac{\partial}{\partial \bar{x}} + b\frac{\partial}{\partial \bar{y}} + c\frac{\partial}{\partial \bar{z}},$$

where $a, b, c, (c \neq 0)$, are smooth functions on V. Now, we consider on V the linearly independent vector fields $\xi, \phi X, W = c\frac{\partial}{\partial \bar{z}}$. An easy calculation implies $\frac{\partial c}{\partial y} = 0, \frac{\partial c}{\partial x} = 0$ and so $[\phi X, W] = [\xi, W] = [\xi, \phi X] = 0$. This means that there exists a chart $\{U, (x, y, z')\}$ at P such that $\xi = \frac{\partial}{\partial x}, \phi X = \frac{\partial}{\partial y}, W = \frac{\partial}{\partial z'}$. On U we have $\xi = \frac{\partial}{\partial \bar{x}} = \frac{\partial}{\partial x}, \phi X = \frac{\partial}{\partial \bar{y}} = \frac{\partial}{\partial y}$ and $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z'}$, as it follows from $W = \frac{\partial}{\partial z'} = c\frac{\partial}{\partial \bar{z}} = X - a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}$. Using it, $X\kappa = 1, \frac{\partial \kappa}{\partial y} = \phi X\kappa = 0$ and $\frac{\partial \kappa}{\partial x} = \xi\kappa = 0$ we get $\frac{\partial \kappa}{\partial z'} = 1$ and so $\kappa = z' + d$, where d is an integration constant. The substitution $z = z' + d$, locally completes the proof of the Lemma in case 1.

Case 2. Let $\mu = 2(1+\lambda)$. Working, as in case 1, for the integrable distribution, which is obtained by ξ and X ($[\xi, X] = 0$), we finally find that there exists a chart (x,y,z) at $P\epsilon M$ on whose domain U, $\kappa = z, \xi = \frac{\partial}{\partial x}, X = \frac{\partial}{\partial y}$ and $\phi X = a'\frac{\partial}{\partial x} + b'\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, where a', b' are smooth functions on M. Since $\lambda = \sqrt{1-\kappa}$, it is obvious that $z < 1$ in both cases.

Now, we will prove that domain U of the above chart can be extended such as to be the whole of M. We will prove case 1, as far as the proof of case 2 is analogous. We suppose that (A, ψ) is a chart at P such that the open set A is the largest possible extension of U. Let $A \neq M$. Then, for any $q\epsilon\partial A$, there exists (as we have proved) a chart $(V, \bar{\psi}(\bar{x}, \bar{y}, \bar{z}))$ at q, such that $\kappa = \bar{z}, \xi = \frac{\partial}{\partial \bar{x}}, \phi X = \frac{\partial}{\partial \bar{y}}$. On $A \cap V$ we have $\bar{z} = z, \xi = \frac{\partial}{\partial \bar{x}} = \frac{\partial}{\partial x}$ and $\phi X = \frac{\partial}{\partial \bar{y}} = \frac{\partial}{\partial y}$. From these, we get $(\bar{x}, \bar{y}, \bar{z}) = (x + c_1, y + c_2, z)$, where c_1, c_2 are integration constants. We consider the smooth function ω on $A \cup V$, such that $\omega = \psi$ on A and $\omega = \bar{\psi} - (c_1, c_2, 0)$ on V. Then, $(A \cup V, \omega)$ defines a new chart of M at P, whose domain $A \cup V \supset A$. By this contradiction, we conclude that $A = M$. This completes the proof of the Lemma.

An immediate and expected consequence of the above result is the following Corollary.

Corollary 6. There are no compact, generalized (κ, μ)-contact metric manifolds with $\|grad\kappa\| = \text{const.} \neq 0$.

Now, we will state and prove our main result.

Theorem 7. Let $M(\eta, \xi, \phi, g)$ be a 3-dimensional generalized (κ, μ)-contact metric manifold with $\|grad\kappa\| = 1$. Then M is covered by a chart $(x, y, z), z < 1$, such that $\kappa = z$ and $\mu = 2(1 - \sqrt{1-z})$ or $\mu = 2(1 + \sqrt{1-z})$. In the first case ($\mu = 2(1 - \sqrt{1-z})$), the following are valid,

$$\xi = \frac{\partial}{\partial x}, \quad \phi X = \frac{\partial}{\partial y} \quad \text{and} \quad X = a\frac{\partial}{\partial x.} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

In the second case ($\mu = 2(1 + \sqrt{1-z})$), the following are valid,

$$\xi = \frac{\partial}{\partial x}, \quad X = \frac{\partial}{\partial y} \quad \text{and} \quad \phi X = a'\frac{\partial}{\partial x} + b'\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

where $a(x,y,z) = -2y + f(z)$, $\quad a'(x,y,z) = 2y + h(z) \quad b(x,y,z) = b'(x,y,z) = 2x\sqrt{1-z} + \frac{y}{4(1-z)} + r(z)$ and f, r, h are smooth functions of z.

Proof. Because of Lemma 5 (see, also its proof) we just have to calculate functions a, b, a', b'.
Let $\mu = 2(1 - \sqrt{1-z})$. Then

$$[\xi, X] = \frac{\partial a}{\partial x}\frac{\partial}{\partial x} + \frac{\partial b}{\partial x}\frac{\partial}{\partial y} \quad \text{and} \quad [X, \phi X] = -\frac{\partial a}{\partial y}\frac{\partial}{\partial x} - \frac{\partial b}{\partial y}\frac{\partial}{\partial y}.$$

Combining these, with $[\xi, X] = 2\lambda\phi X = 2\lambda\frac{\partial}{\partial y}$ and $[X, \phi X] = -\frac{1}{4\lambda^2}\phi X + 2\xi = -\frac{1}{4\lambda^2}\frac{\partial}{\partial y} + 2\frac{\partial}{\partial x}$ (see, Lemma 3), we get

$$\frac{\partial a}{\partial x} = 0, \quad \frac{\partial b}{\partial x} = 2\lambda, \quad \frac{\partial a}{\partial y} = -2 \quad \text{and} \quad \frac{\partial b}{\partial y} = \frac{1}{4\lambda^2}.$$

It follows from this system that, $a = -2y + f(z)$ and $b = 2x\sqrt{1-z} + \frac{y}{4(1-z)} + r(z)$, where $f(z), r(z)$ are integration functions.
Now, let $\mu = 2(1 + \sqrt{1-z})$. We have

$$[\xi, \phi X] = \frac{\partial a'}{\partial x}\frac{\partial}{\partial x} + \frac{\partial b'}{\partial x}\frac{\partial}{\partial y} \quad \text{and} \quad [X, \phi X] = \frac{\partial a'}{\partial y}\frac{\partial}{\partial x} + \frac{\partial b'}{\partial y}\frac{\partial}{\partial y}.$$

Combining these, with $[\xi, \phi X] = 2\lambda X = 2\lambda\frac{\partial}{\partial y}$ and $[X, \phi X] = \frac{1}{4\lambda^2}X + 2\xi = \frac{1}{4\lambda^2}\frac{\partial}{\partial y} + 2\frac{\partial}{\partial x}$ we get

$$\frac{\partial a'}{\partial x} = 0, \quad \frac{\partial b'}{\partial x} = 2\lambda, \quad \frac{\partial a'}{\partial y} = 2 \quad \text{and} \quad \frac{\partial b'}{\partial y} = \frac{1}{4\lambda^2}$$

and so $a'(x, y, z) = 2y + h(z)$ and $b'(x, y, z) = 2x\sqrt{1-z} + \frac{y}{4(1-z)} + r(z)$, where $h(z), p(z)$ are integration functions. This competes the proof of the Theorem.

Remark 3. The functions a, b, a', b' of Theorem 7 determine the manifold completely, as we will see in the next paragraph. There, using the conclusion of Lemma 5, we will construct in R^3 all the generalized (κ, μ)-c.m.m. with $\|grad\kappa\| = 1$.

5. Construction. Let $M = \{(x, y, z)\epsilon R^3/z < 1\}$ and $f, r : M \to R$ be arbitrary functions of z. We consider the linearly independent vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial y},$$

where $a(x, y, z) = -2y + f(z), b(x, y, z) = 2x\sqrt{1-z} + \frac{y}{4(1-z)} + r(z)$. Let g be the Riemannian metric defined by $g(e_i, e_j) = \delta_{ij}, (i, j = 1, 2, 3)$, ∇ the Riemannian connection and R the curvature tensor of g. Putting $\lambda = \sqrt{1-z}$, we easily get $[e_1, e_3] = 0, [e_1, e_2] = 2\lambda e_3, [e_2, e_3] = -\frac{1}{4\lambda^2}e_3 + 2e_1$. Moreover, we define the 1-form η and the (1,1)-tensor field ϕ by $\eta(.) = g(., e_1)$ and $\phi e_1 = 0, \phi e_2 = e_3, \phi e_3 = -e_2$. Because $\eta \wedge d\eta \neq 0$ everywhere on M, η is a contact form. Using the linearity of ϕ, $d\eta$ and g we find $\eta(e_1) = 1, \phi^2 Z = -Z + \eta(Z)e_1, d\eta(Z, W) = g(Z, \phi W)$ and $g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W)$ for any $Z, W\epsilon\mathcal{X}(\mathcal{M})$. Hence $M(\eta, e_1, \phi, g)$ defines a contact metric structure on M. Putting $\xi = e_1, X = e_2, \phi X = e_3$ and using the well known formula

$$\begin{aligned} 2g(\nabla_Y Z, W) &= Yg(Z, W) + Zg(W, Y) - Wg(Y, Z) \\ &\quad -g(Y, [Z, W]) - g(Z, [Y, W]) + g(W, [Y, Z]), \end{aligned}$$

we find the formulas (9)-(13). Moreover, for the tensor field h we get $h\xi = 0, hX = \lambda X, h\phi X = -\lambda\phi X$. Using the above relations and the definition of the curvature tensor, we finally get that $M(\eta, \xi, \phi, g)$ is a generalized (κ, μ)-c.m.m. (of type A) with $\kappa = z$ and $\mu = 2(1 - \sqrt{1-z})$.

In order to construct an arbitrary generalized (κ, μ)-c.m.m. with $\|grad\kappa\| = 1$ of type B, we work analogously on the same manifold M, considering the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} \quad e_3 = a'\frac{\partial}{\partial x} + b'\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

where $a' = 2y + f(z)$, $b' = 2x\sqrt{1-z} + \frac{y}{4(1-z)} + r(z)$. The tensor fields g, η, ϕ are defined by $g(e_i, e_j) = \delta_{ij}, (i, j = 1, 2, 3)$, $\eta(.) = g(., e_1)$, $\phi e_1 = 0, \phi e_2 = e_3$ and $\phi e_3 = -e_2$. Putting $\xi = e_1, X = e_2$ and $\phi X = e_3$ we finally find that $M(\eta, \xi, \phi, g)$ is a generalized (κ, μ)-c.m.m. (of type B), with $\kappa = z$ and $\mu = 2(1 + \sqrt{1-z})$.

Remark 4. The examples 1 and 2 of §3 correspond in the special case $f = 0, r = 0$.

In §3 we have seen that a D_a-homothetic deformation tranforms a generalized (κ, μ)-c.m.m. with $\|grad\kappa\| = 1$ to another generalized $(\bar{\kappa}, \bar{\mu})$-c.m.m. with $\|grad\bar{\kappa}\| = d \neq 1$(const.). In the next paragraph we will introduce a second transformation, which transforms a generalized (κ, μ)-c.m.m. $M(\eta, \xi, \phi, g)$ with $\|grad\kappa\| = 1$ to another generalized (κ, μ)-c.m.m. $M(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ with the same κ, μ, $\|grad\kappa\|_{\bar{g}} = 1$ and of the same type.

6. Another transformation. Let $M(\eta, \xi, \phi, g)$ be a generalized (κ, μ)-c.m.m. with $\|grad\kappa\| = 1$, and f, r smooth functions on M such that $\xi f = \xi r = 0$ and $(\phi grad\kappa)f = (\phi grad\kappa)r = 0$. We consider the vector fields

$$\bar{\xi} = \xi, \quad \bar{X} = grad\kappa + f\xi + r(\phi grad\kappa), \quad \bar{Y} = \phi grad\kappa$$

and we define the tensor fields $\bar{g}, \bar{\eta}, \bar{\phi}$ as follows,

$$\bar{g}(\bar{\xi}, \bar{\xi}) = \bar{g}(\bar{X}, \bar{X}) = \bar{g}(\bar{Y}, \bar{Y}) = 1, \bar{g}(\bar{\xi}, \bar{X}) = \bar{g}(\bar{\xi}, \bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) = 0$$
$$\bar{\eta}(.) = \bar{g}(., \bar{\xi}), \quad \bar{\phi}\bar{\xi} = 0, \bar{\phi}\bar{X} = \bar{Y}, \bar{\phi}\bar{Y} = -\bar{X}.$$

Then $M(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ is a generalized (κ, μ)-c.m.m. with $\|grad\kappa\|_{\bar{g}} = 1$ (with the same κ, μ).

To prove it, we distinguish two cases: $\mu = 2(1-\lambda)$ and $\mu = 2(1+\lambda)$. We will prove the first case, because the proof of the second case is similar. Let $\mu = 2(1-\lambda)$. Then, as we have seen in Lemma 3, $\xi\kappa = 0, (grad\kappa)\kappa = 1, (\phi grad\kappa)\kappa = 0$. Therefore, there exists a global coordinate system (x, y, z), (see, Theorem 7 and its proof) such that , $\kappa = z, \xi = \frac{\partial}{\partial x}, \phi grad\kappa = \frac{\partial}{\partial y}$ and $grad\kappa = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, where $a = -2y + h(z), b = 2x\sqrt{1-z} + \frac{y}{4(1-z)} + \nu(z)$ and h, ν smooth functions of z. Then $\bar{X} = (-2y + F(z))\frac{\partial}{\partial x} + (2x\sqrt{1-z} + \frac{y}{4(1-z)} + G(z))\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ and $\bar{Y} = \frac{\partial}{\partial y}$, where $F = f + h$ and $G = r + \nu$. According to the construction of §5, $M(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ is also a generalized (κ, μ)-c.m.m. with $\kappa = z$ and $\mu = 2(1 - \sqrt{1-z})$.

Remark 5. Any generalized (κ, μ)-c.m.m. with $||grad\kappa||$ =const. $\neq 0$ can be obtained by examples 1 and 2 of §3, under the above transformation and a D_a-homothetic deformation.

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